Elliptic PDEs
LECTURE 1

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Prerequisites Part III Analysis of PDEs.

Reading: • Gilbarg & Trudinger "Elliptic PDEs of 2nd order".

[Paper] • L. Simon "Schauder estimates by scaling" Calc. Var & PDE 5, 1997 pp. 391-407.

[old lecture notes] • minterscompactness.wordpress.com/lecture-notes/ (Paul Minter's page).

[Broader reading] • Folland "Introduction to PDEs".
• Evans & Gariepy "Measure Theory & Fine Properties of Functions".

§0 Introduction

We study 2nd order elliptic PDEs on (a domain in) $\mathbb{R}^n$, as e.g. arising from variational problems. Ultimately: nonlinear PDEs. First understand linear theory.

Setup: Consider for $\Omega \subset \mathbb{R}^n$ open, bounded
$$F: \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$
$$(x, z, p) \longmapsto F(x,z,p).$$
& consider the variational problem:
$$\mathcal{F}[u] := \int_\Omega F(x, u(x), \partial u(x))\, dx \qquad \partial = D = \nabla$$
and assume F is sufficiently regular.
Let $u \in S$ a suitable vector space of functions $u: \Omega \to \mathbb{R}$ (frequently $S = H^1(\Omega) = \{f \in L^2(\Omega) : \nabla f \in L^2(\Omega)\}$ or $S \subseteq C^{1,\alpha}(\Omega)$ (later)).
Suppose that u minimises $\mathcal{F}[u]$ subject to* $u|_{\partial\Omega} = g$ for some given $g: \partial\Omega \to \mathbb{R}$. So $\forall \varphi \in S$ $\mathcal{F}[u + t\varphi] \geq \mathcal{F}[u]$

* this tends to be needed for well-posedness. This means that
$$\frac{d}{dt}\Big|_{t=0} \mathcal{F}[u + t\varphi] = 0$$
or
$$\frac{d}{dt}\Big|_{t=0} \int_\Omega F(x, u + t\varphi, \partial u + t\partial\varphi)\, dx = 0.$$
Assuming enough regularity to exchange $\frac{d}{dt}$ & $\int$, get
$$\int_\Omega (\partial_z F)(x, u, \partial u)\varphi + \partial_i \varphi (\partial_{p_i} F)(x, u, \partial u)\, dx = 0. \quad (0.1)$$
To ensure the perturbed $u + t\varphi$ has correct BC, need $\varphi|_{\partial\Omega} = 0$. So integrate (0.1) by parts:
$$\int_\Omega \varphi(x)(\partial_z F - \partial_i \partial_{p_i} F)(x, u, \partial u)\, dx = 0$$
$\forall \varphi \in S$ & so (Fundamental Lemma of Calc of Var's)
$$\Rightarrow \frac{\partial F}{\partial z} - \frac{\partial}{\partial x^i}\left(\frac{\partial F}{\partial p_i}\right) = 0 \text{ in } \Omega$$
– Euler–Lagrange eqn for F ($\mathcal{F}$)
Can rewrite this as
$$(0.2) \quad \frac{\partial F}{\partial z} - \partial_i \partial_j u \frac{\partial^2 F}{\partial p_i \partial p_j} = 0$$
– a 2nd order quasilinear PDE in u.
↳ means the term in front of $\partial^2 u$ does not depend on $\partial^2 u$.

More generally, consider
$$a^{ij}(x, u, \partial u)\partial^2_{ij} u - b(x, u, \partial u) = 0 \quad (0.3)$$

Definition We say (0.3) is elliptic in Ω if $a^{ij}(x, u, \partial u)$ is a positive-definite matrix in Ω.
In the case (0.2), this is then equivalent to "F is convex in p."

Example (Dirichlet energy) When $F(x,z,p) = |p|^2$, one gets $\Delta u = 0$. (0.4)
Extremizers of (0.4) are called harmonic functions.

Example (Minimal surfaces). When $F(x,z,p) = \sqrt{1+|p|^2}$ (exercise: interpret $\mathcal{F}[u]$), one gets
$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0. \quad (0.5)$$
– the minimal surface equation

Remark: Locally $\nabla u \sim$ constant, so (0.5) looks similar to (0.4), & so solutions have similar local properties. But the existence theory for (0.4) is "trivial", while the existence theory for (0.5) may fail. (Global properties are important!) For entire solutions (i.e. defined on all of $\mathbb{R}^n$), global behaviour very different:
Thm (Liouville) If $u: \mathbb{R}^n \to \mathbb{R}$, $u \in C^2$, $\Delta u = 0$ and u is bounded, then $u \equiv$ const.
Thm (Bernstein) [The only entire solutions to (0.5) in $\mathbb{R}^n$ are planar (u is linear)] $\iff n \leq 7$.

§1 Harmonic Functions

1.1 Basic Properties

Let $\Omega \subset \mathbb{R}^n$ be a domain (open & connected).
Definition: A function $u \in C^2(\Omega)$ is harmonic if $\Delta u = 0$, subharmonic if $\Delta u \geq 0$, superharmonic if $\Delta u \leq 0$ in Ω.

Write $B_\rho(y) = \{x : |x - y| < \rho\}$

Theorem 1.2 (Mean Value Property (MVP))
If $u \in C^2(\Omega)$ is subharmonic and $B_r(y) \subset \Omega$, then
$$(1.1) \quad u(y) \leq \frac{1}{\omega_n \cdot r^n}\int_{B_r(y)} u(x)\, dx, \quad \omega_n = |B_1(0)|$$
&
$$(1.2) \quad u(y) \leq \frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r(y)} u(x)\, dx$$
If u is superharmonic, then the inequalities are reversed. If harmonic, equalities.

Proof: We have
$$0 \leq \int_{B_\rho(y)} \Delta u\, dx \overset{IBP}{=} \int_{\partial B_\rho(y)} \nabla u \cdot \omega\, dx \qquad \text{outward unit normal } \omega = \frac{x-y}{\rho}$$
$$\overset{\rho\omega = x-y}{=} \rho^{n-1}\int_{S^{n-1}} \omega \cdot \nabla u(y + \rho\omega)\, d\omega.$$
$$= \rho^{n-1}\int_{S^{n-1}} \frac{\partial}{\partial\rho}(u(y + \rho\omega))\, d\omega$$
This is true $\forall \rho$, so
$$0 \leq \frac{\partial}{\partial\rho}\int_{S^{n-1}} u(y + \rho\omega)\, d\omega$$
i.e. the map $\rho \mapsto \int_{S^{n-1}} u(y + \rho\omega)\, d\omega$ is increasing,
i.e. for $\rho \leq r$ $\int_{S^{n-1}} u(y + \rho\omega)\, d\omega \leq \int_{S^{n-1}} u(y + r\omega)\, d\omega$
for $0 \leq \rho \leq r$. By continuity, let $\rho \to 0$, to get $n \cdot \omega_n u(y) \leq \frac{1}{r^{n-1}}\int_{\partial B_r(y)} u(x)\, dx$. This gives (1.2).
To get (1.1), integrate in r. The superharmonic case is similar & the harmonic case combines both. □

Remark: The MVP characterises harmonic functions (Sheet 1).

ELLIPTIC PDES
LECTURE 2

Last time: $u \in C^2(\Omega)$ harmonic $\iff u$ satisfies MVP.

L2

Theorem 1.3 (Strong Maximum Principle)

Suppose $u \in C^2(\Omega)$ is subharmonic on Ω ($\Delta u \geq 0$), and suppose $\exists y_0 \in \Omega$ s.t. $u(y_0) = \sup_\Omega u$. Then $u \equiv$ const.

Rmk: if u is superharmonic, then same statement holds with "sup" $\to$ "inf". If u harmonic, both work.

Proof: Let $M = \sup_\Omega u < \infty$ and let $\Sigma = \{y \in \Omega : u(y) = M\}$. By assumption, $\Sigma \neq \emptyset$ since $y_0 \in \Sigma$, and Σ is closed as u is continuous. As Ω is connected, it suffices to show that Σ is open. Then $\Sigma = \Omega$.

Pick $y \in \Sigma$. By the MVP for $\rho > 0$ s.t. $B_\rho(y) \subset \Omega$. Have
$$M = u(y) \leq \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} u(x)\,dx$$

so
$$\frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} (M - u(x))\,dx \leq 0.$$

But of course, $M - u \geq 0$, so must have $u \equiv M$ on $B_\rho(y)$. So Σ open. □

Here the SMP is easy given the MVP. For more general PDEs, this is not so. We prove a weaker statement first.

Theorem 1.4 (Weak Maximum Principle (WMP))

Suppose $\Omega \subset \mathbb{R}^n$ is bounded and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If u is subharmonic on Ω, then $\sup_\Omega u = \sup_{\partial\Omega} u$.

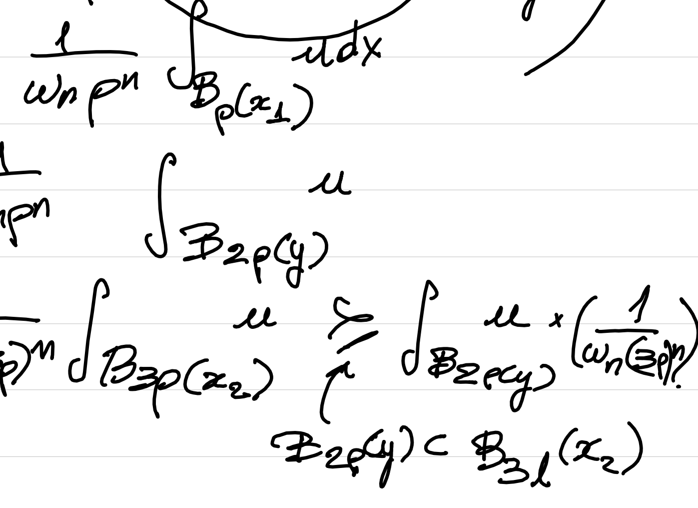
Proof: Immediate from SMP: since Ω is bounded and $u \in C(\overline{\Omega}) \Rightarrow \sup_{\overline{\Omega}} u$ & $\inf_{\overline{\Omega}} u$ are attained. By the SMP, these are not attained in Ω° (unless u is constant). □

Remark: if u is superharmonic, replace "sup" with "inf." If u is harmonic, both hold.

MVP states that u always an average of itself. Suggests that u cannot vary too much.
Q: can use this to relate $\sup u$ & $\inf u$?
Yes

Theorem 1.5 (Harnack's Inequality)
Suppose $u \in C^2(\Omega)$, $u \geq 0$ and $\Delta u = 0$ in Ω. Then if $\Omega' \subset\subset \Omega$ is any bdded subdomain, we have $\sup_{\Omega'} u \leq C \cdot \inf_{\Omega'} u$ for some $C = C(n, \Omega', \Omega)$ (indep. of u).

Proof: First, choose $y \in \Omega$ and $\rho > 0$ s.t. $B_{4\rho}(y) \subset \Omega$ and pick $x_1, x_2 \in B_\rho(y)$.

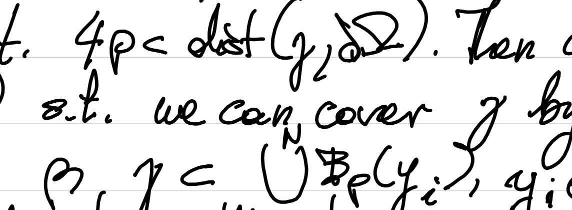


$$\text{MVP} \Rightarrow u(x_1) = \frac{1}{\omega_n \rho^n} \int_{B_\rho(x_1)} u\,dx$$
$B_\rho(x_1) \subset B_{2\rho}(y)$
$$\leq \frac{1}{\omega_n \rho^n} \int_{B_{2\rho}(y)} u$$

& $$u(x_2) = \frac{1}{\omega_n (3\rho)^n} \int_{B_{3\rho}(x_2)} u \geq \int_{B_{2\rho}(y)} u \times \left(\frac{1}{\omega_n (3\rho)^n}\right)$$
$B_{2\rho}(y) \subset B_{3\rho}(x_2)$

Combining these: $u(x_1) \leq 3^n u(x_2)$ $\forall x_1, x_2 \in B_\rho(y)$.
So Harnack holds locally in balls with constant indep. of u, ρ, y as long as $\overline{B_{4\rho}(y)} \subset \Omega$.
So now, choose $x_1, x_2 \in \overline{\Omega'} \subset \Omega$, s.t. $\sup_{\Omega'} u = u(x_1)$ & $u(x_2) = \inf_{\Omega'} u$.

Then by path connectedness of Ω, $\exists$ a cts path $\gamma \in \overline{\Omega'}$ joining x_1 & x_2.



$4\rho < \mathrm{dist}(\overline{\Omega'}, \partial\Omega) \leq \mathrm{dist}(\gamma, \partial\Omega)$

Choose $\rho > 0$ s.t. $4\rho < \mathrm{dist}(\gamma, \partial\Omega)$. Then choose $N = N(\Omega', \Omega)$ s.t. we can cover γ by N balls of radius ρ, $\gamma \subset \bigcup_{i=1}^N B_\rho(y_i)$, $y_i \in \Omega'$. Then apply the local result along each ball to get $u(x_1) \leq 3^n \times 3^n \times \cdots \times 3^n\, u(x_2)$
$$\leq 3^{nN} u(x_2)$$ □

Theorem 1.6 (Derivative Estimates)

Suppose $u \in C^3(\Omega)$ is harmonic in Ω. Then if $\overline{B_\rho(y)} \subset \Omega$, then
$$|Du(y)| \leq \frac{C}{\rho} \sup_{\partial B_\rho(y)} |u|$$
for some $C = C(n)$.

Proof: $\Delta u = 0 \Rightarrow 0 = D_i(\Delta u) = \Delta(D_i u)$ in Ω. So $D_i u$ is harmonic. By the MVP,
$$D_i u(y) = \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} D_i u$$
$$= \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} \nabla \cdot (u e_i)$$
$$= \frac{1}{\omega_n \rho^n} \int_{\partial B_\rho(y)} u \cdot \underbrace{n_i}_{\frac{x_i - y_i}{\rho}}\,ds$$

So $$|D_i u(y)| \leq \frac{n}{\omega_n \rho^n} \sup_{\partial B_\rho(y)} |u| \cdot \int_{\partial B_\rho(y)} ds$$
$$= \frac{n}{\rho} \sup_{\partial B_\rho(y)} |u|$$ □

Remark: Can apply this result repeatedly to get that for $\Omega' \subset \Omega'' \subset \Omega$ & any multi-index $|\alpha| \leq k$ if $u \in C^{k+2}(\Omega)$, $\Delta u = 0$ in Ω, then $\sup_{\Omega'} |D^\alpha u| \leq C \cdot \sup_{\Omega''} |u|$ for some $C = C(n, \alpha, \Omega, \Omega')$.
(i.e. $\|D^\alpha u\|_{L^\infty(\Omega')} \leq C \cdot \|u\|_{L^\infty(\Omega'')}$).
Also by the MVP for some $y \in \overline{\Omega''} \subset \Omega$
$$\sup_{\Omega''} |u| = |u(y)| = \left| c \int_{B_\rho(y)} u(x)\,dx \right|$$
$$\leq C \int_\Omega |u|$$
(i.e. $\|D^\alpha u\|_{L^\infty(\Omega')} \leq C \cdot \|u\|_{L^1(\Omega)}$).

Theorem 1.7 (Uniqueness of Solutions to Dirichlet Problem).
Suppose that Ω is bounded & $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ & $\begin{cases} \Delta u_1 = \Delta u_2 & \text{in } \Omega \\ u_1 = u_2 & \text{on } \partial\Omega \end{cases}$
Then $u_1 \equiv u_2$ on $\overline{\Omega}$.

Proof: Set $w = u_1 - u_2$. Then $\Delta w = 0$ in Ω, and $w = 0$ on $\partial\Omega$. By applying WMP, get $w = 0$ in Ω. □

Remark: Can of course integrate by parts here but WMP will apply for non-divergence form equations.

Elliptic PDEs
Lecture 3

Last time, for $\Omega' \subset\subset \Omega$

$$\sup_{\Omega'} |D^\alpha u| \le C \cdot \int_\Omega |u|$$

Theorem 1.8 (Liouville's Theorem for Harmonic Functions).
$\forall u \in C^\infty(\mathbb{R}^n)$ is harmonic in $\mathbb{R}^n$ and grows sublinearly at ∞, then $u = \text{const}$.

Remark: "Growing sublinearly" means
$$|u(x)| \le C \cdot (1 + |x|^\alpha), \quad \alpha \in (0,1).$$

Proof From derivative estimates (Thm 1.6) we know that $\forall y \in \mathbb{R}^n$
$$|Du(y)| \le \frac{C}{\rho} \sup_{B_\rho(y)} |u|$$
Plug in growth assumption:
$$|Du(y)| \le \frac{C}{\rho} \sup_{B_\rho(y)} |u| \le \frac{C}{\rho}\left(1 + (\rho + |y|)^\alpha\right)$$
Take $\rho \to \infty$ to get $Du(y) = 0$. But y was arbitrary, so we are done. $\square$

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§1.2. Existence Theory for Harmonic Functions

Classical problem: solve the Dirichlet problem for the Laplacian on Ω bounded and $\varphi : \overline{\Omega} \to \mathbb{R}$ continuous, we wish to find $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ s.t.
$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$
We will assume for simplicity that $\partial\Omega$ is smooth & $\varphi \in C^\infty(\overline{\Omega}, \mathbb{R})$.

Methods to solve the problem:

(I) **Hilbert Space Method:** Use the Riesz representation theorem to get $u \in H^1(\Omega)$. Then deal with regularity afterwards. Relies on the equation being linear (cf Analysis of PDE)

(II) **Direct Method of Calculus of Variations**
Rephrase $\Delta u = 0$ as a variational problem (i.e. the Euler-Lagrange equation of $\int |\nabla u|^2$) and prove existence using the functional.

(III) **Perron's Method** Use the fact that solvability in balls implies solvability in more general domains. More later. Based on maximum principles.

Remark In all cases we obtain a regular solution first, and improve regularity later.

Look at (II) in detail. Define
$$\mathcal{L} = \{ w \in H^1(\Omega) : w - \varphi \in H^1_0(\Omega) \}$$
i.e. H^1 functions which agree with φ on the boundary. Check $\varphi \in \mathcal{L}$, so $\mathcal{L} \neq \emptyset$. Set
$$\mathcal{E}[w] = \int_\Omega |Dw|^2.$$
and define $\beta = \inf_{w \in \mathcal{L}} \mathcal{E}[w]$

By def^n of inf, $\exists (w_j) \subset \mathcal{L}$ s.t. $\mathcal{E}[w_j] \to \beta$. We want to extract a convergent subsequence and show that its limit is a sol^n. Clearly for j large
$$\int_\Omega |Dw_j|^2 \le \beta + 1.$$
Since $w_j - \varphi \in H^1_0(\Omega)$. By the Poincaré inequality,
$$\int_\Omega |w_j - \varphi|^2 \le C \cdot \int_\Omega |D(w_j - \varphi)|^2$$
$$\Rightarrow \int_\Omega |w_j|^2 \le C(\Omega, \varphi, \beta) < \infty$$
Indeed,
$$\|w_j - \varphi\|^2_{L^2(\Omega)} \le C \cdot \|D(w_j - \varphi)\|^2_{L^2(\Omega)} \le C(\Omega, \varphi, \beta) < \infty.$$
$$\|w_j\|^2 - 2\langle w_j, \varphi \rangle_{L^2(\Omega)} \le C(\Omega, \varphi, \beta)$$
$$\Rightarrow \|w_j\|^2_{L^2(\Omega)} \le C(\varphi, \Omega, \beta) + \varepsilon \|w_j\|^2_{L^2} + 1/\varepsilon \|\varphi\|^2_{L^2}$$
$$\Rightarrow \|w_j\|^2_{L^2(\Omega)} \le C(\varphi, \Omega, \beta)$$
So we have $\|w_j\|^2_{H^1(\Omega)} \le C$ for j large, so by Banach-Alaoglu
$$w_{j_k} \rightharpoonup w \text{ in } H^1(\Omega)$$
and by Rellich-Kondrachov
$$w_{j_k} \to w \text{ in } L^2(\Omega)$$
for some $w \in H^1(\Omega)$.

Digression 1
Rellich-Kondrachov Ω bdded, $1 \le p < n$
$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$
and
$$W^{1,p}(\Omega) \subset\subset L^q(\Omega) \quad 1 \le q < p^*,$$
where $p^* = \frac{np}{n-p}$. When $p = 2$,
$$p^* = \frac{2n}{n-2} > 2 = q \text{ iff } n > 2$$

Hence $\forall v \in H^1(\Omega)$ have
$$\int_\Omega Dw_j \cdot Dv \to \int_\Omega Dw \cdot Dv$$
Also clearly that $w_{j_k} - \varphi \rightharpoonup w - \varphi$ in $H^1(\Omega)$ as φ is smooth. But $w_{j_k} - \varphi \in H^1_0(\Omega)$ and $H^1_0(\Omega)$ is norm-closed, hence weakly closed

Digression 2 This follows from Hahn-Banach for any convex subset of a Banach space

Lemma X a Banach space. Then if $C \subset X$ a convex subset then C is norm-closed $\Leftrightarrow$ C is weakly closed

Proof: $\Leftarrow$) Exercise
$\Rightarrow$) We show $X \setminus C$ is weakly open. Let $x_0 \in X \setminus C$. By Hahn-Banach Separation, $\exists \phi \in X'$ such that $\phi|_C = 0$ and $\phi(x_0) \neq 0$. Then $\{ x \in X : |\phi(x)| > 1/2 \cdot |\phi(x_0)| \} \subset X \setminus C$ is weakly open. $\square$

Hence $w - \varphi \in H^1_0(\Omega)$ i.e. $w \in \mathcal{L}$. Finally, since $\mathcal{E}[\cdot]$ is sequentially weakly lower semicontinuous in $H^1(\Omega)$, we have:
$$\mathcal{E}[w] \le \liminf_{k \to \infty} \mathcal{E}[w_{j_k}] = \beta$$
so $\mathcal{E}[w] = \beta$.

Digression 3
$\mathcal{E}[\cdot]$ is sequentially weakly lower semi-continuous means $\forall u_j \rightharpoonup u$ in $H^1(\Omega)$, $\mathcal{E}[u] \le \liminf_{j \to \infty} \mathcal{E}[u_j]$.
To see this, note
$$\int_\Omega Du_j \cdot Dv \to \int_\Omega Du \cdot Dv$$
so with $v = u$
$$\int_\Omega Du_j \cdot Du \to \int_\Omega |Du|^2$$
So
$$\mathcal{E}[u] = \lim_j \int_\Omega Du_j \cdot Du = \liminf_j \int_\Omega Du_j \cdot Du \le \liminf_j \mathcal{E}[u_j]^{1/2} \cdot \mathcal{E}[u]^{1/2}$$

We have found a global min w, i.e. $\forall v \in H^1_0(\Omega)$, $w + tv \in \mathcal{L}$, $\mathcal{E}[w + tv] \ge \mathcal{E}[w]$. i.e. the derivative of $\mathcal{E}[w + tv]$ at $t = 0$ vanishes.
$$f'(0) = D\mathcal{E}[u](v) = \lim_{t \to 0} \frac{\mathcal{E}[w + tv] - \mathcal{E}[u]}{t}$$
$$= 2 \int_\Omega Dw \cdot Dv = 0 \quad \forall v \in H^1_0(\Omega).$$
This is the weak formulation of $\Delta w = 0$.

Next time: regularity theory (weak solutions of $\Delta w = 0$ are smooth).

Elliptic PDEs
Lecture 4

§1.3: Interior Regularity

We wish to prove more regularity for the weak solution. We have shown $\exists u \in L^1(\Omega)$ s.t. ($\hookrightarrow$ bdd)

$$\int_\Omega u \Delta v = 0 \qquad \forall v \in C_c^\infty(\Omega)$$

Key result.

Theorem 1.9: (Weyl's Lemma)
Weakly harmonic functions are smooth. That is, for $\Omega \subset \mathbb{R}^n$ open and $u \in L^1_{loc}(\Omega)$, if $\int_\Omega u \Delta v = 0$ $\forall v \in C_c^\infty(\Omega)$ then $u \in C^\infty(\Omega)$ & $\Delta u = 0$.

Proof: Mollify u: take $\varphi \in C^\infty(\mathbb{R}^n)$ s.t. $\varphi \equiv 0$ in $\mathbb{R}^n \setminus B_1(0)$, $\varphi \geq 0$

& $\int_{\mathbb{R}^n} \varphi = 1$.

WLOG take φ to be radially symmetric. For $\sigma > 0$ put $\varphi_\sigma(x) = \frac{1}{\sigma^n} \varphi(x/\sigma)$

Then $\varphi_\sigma \in C_c^\infty(B_\sigma(0))$, $\varphi_\sigma \geq 0$ & $\int_{\mathbb{R}^n} \varphi_\sigma = 1$. This is the "standard mollifier".
Define $u_\sigma(x) = (\varphi_\sigma * u)(x)$.
Then this is well-defined for $x \in \Omega_\sigma = \{x \in \Omega : dist(x, \partial\Omega) > \sigma\}$ (u only defined in Ω). Then u_σ is smooth in Ω_σ and $u_\sigma \to u$ in $L^1_{loc}(\Omega)$

(Thm 4.1 in Evans & Gariepy) & also $\Delta u_\sigma = 0$.

$$\frac{\partial}{\partial x_i} u_\sigma(x) = \int_\Omega u(y) \frac{\partial}{\partial x_i} \varphi_\sigma(x-y)\,dy$$

$$= -\int_\Omega u(y) \frac{\partial}{\partial y_i} \varphi_\sigma(x-y)\,dy$$

$$\Rightarrow \quad \Delta_x u_\sigma(x) = \int_\Omega u(y) \Delta_y \varphi_\sigma(x-y)\,dy$$

$$= 0.$$

By a priori derivative estimates for harmonic functions, for $\Omega' \subset\subset \Omega$

$$\sup_{\Omega'} |D^\alpha u_\sigma| \leq C \cdot \int_{\Omega'_{\sigma_1}(\Omega')} |u_\sigma|$$

for some $\sigma_1(\Omega')$ small, where $\Omega'_{\sigma} = \Omega' \cup \{x \in \Omega : dist(x, \partial\Omega') < \sigma_1\}$. But $u_\sigma \to u$ in $L^1_{loc}(\Omega)$ so for small enough σ_1,

$$\int_{\Omega'_{\sigma_1}} |u_\sigma| \leq C \left(\int_{\Omega'_{\sigma_1}} |u| + 1 \right)$$

Hence $\sup_{\Omega'} |D^\alpha u_\sigma| \leq C \cdot \left(\int_{\Omega'_{\sigma_1}} |u| + 1 \right)$

i.e. $D^\alpha u_\sigma$ uniformly bounded in $L^\infty(\Omega')$.
Hence (as bounded derivs $\Rightarrow$ equicontinuous), by Arzelà-Ascoli $\exists (\sigma_j)_{j=1}^\infty$, $\sigma_j \downarrow 0$ & $\exists$ $\tilde{u} \in C^\infty(\Omega)$ s.t. $u_{\sigma_j} \xrightarrow{j} \tilde{u}$ in $C^k(\Omega')$ $\forall k \in \mathbb{N}$. Hence, $\Delta \tilde{u} \overset{?}{=} \lim_{j \to \infty} \Delta u_{\sigma_j} = 0$ in Ω.

as Ω' was arbitrary. But also, $u_\sigma \to u$ a.e. in Ω (properties of mollifications) & so $u = \tilde{u}$ a.e.

Remark: We do not say anything about boundary regularity. But it is possible to get at least $u \in C^\infty(\overline{\Omega})$.

Let's now improve our $C^0(\partial\Omega)$ existence result to $C^0(\partial\Omega)$.

Theorem 1.10 (Existence & Uniqueness for the Dirichlet Problem with $C^0(\partial\Omega)$ data).
Suppose Ω is bounded with sufficiently regular* boundary $\partial\Omega$. Then for any $\varphi \in C^0(\partial\Omega)$ $\exists! u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ solving

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

Remark: We might have $\int_\Omega |Du|^2 = \infty$ for this solution!

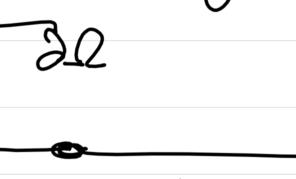
Proof: Choose a sequence $(\varphi_n)_n \subset C^\infty(\mathbb{R}^n)$ s.t. $\varphi_n \to \varphi$ on $\partial\Omega$ uniformly. Then, we know $\exists u_n \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ s.t. $\Delta u_n = 0$ & $u_n = \varphi_n$ on $\partial\Omega$. Then $\forall n, m \in \mathbb{N}$ $\Delta(u_n - u_m) = 0$ in Ω, $u_n - u_m = \varphi_n - \varphi_m$ on $\partial\Omega$.
By the WMP,

$$\sup_{\overline{\Omega}} |u_n - u_m| \leq \sup_{\partial\Omega} |u_n - u_m|$$

$$= \sup_{\partial\Omega} |\varphi_n - \varphi_m| \to 0.$$

as $n, m \to \infty$. So $(u_n)_n$ is Cauchy in $C^0(\overline{\Omega})$. By completeness of $C^0(\overline{\Omega})$, there exists $u \in C^0(\overline{\Omega})$ s.t. $u_n \to u$ uniformly on $\overline{\Omega}$. In particular $u = \varphi$ on $\partial\Omega$.
Further, by the derivative estimates for $(u_n)_n$ also converges in $C^k(\Omega')$ $\forall \Omega' \subset\subset \Omega$ $\forall k \in \mathbb{N}$, so $u \in C^\infty(\Omega)$ & $\Delta u = 0$ $\square$

Remarks: • if $\partial\Omega$ is C^2, then * is satisfied. More generally, enough to have exterior sphere condition: $\forall z \in \partial\Omega$ $\exists B_\rho(y) : \overline{B_\rho(y)} \cap \overline{\Omega} = \{z\}$



• $\exists$ bounded domains in which this fails (& the conclusion of the thm fails), e.g. when $\partial\Omega$ has a cusp.

Now move away from harmonic functions & consider more general

§2. General 2nd Order Elliptic Operators

From now on, write

$$Lu = a^{ij}\partial_i\partial_j u + b^i \partial_i u + cu.$$

work on $\Omega \subset \mathbb{R}^n$ open and $u \in C^2(\Omega)$, $a^{ij}, b^i, c : \Omega \to \mathbb{R}$ & consider the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

for given $f : \Omega \to \mathbb{R}$ & $\varphi : \partial\Omega \to \mathbb{R}$ if we can write L in divergence form, $Lu = \partial_i(a^{ij}\partial_j u) + \tilde{b}^i \partial_i u + cu$ then can use Hilbert space theory. If not, have to use Schauder theory.
Idea is to deform L into Δ using a series of rescalings (does not involve Sobolev spaces). Since $u \in C^2(\Omega)$ let's assume $a^{ij} = a^{ji}$.

Definition 2.1
(I) L is elliptic in Ω if the matrix $a^{ij}(x)$ is positive definite in Ω.
That is $0 < \lambda(x) \cdot |\xi|^2 \leq a^{ij}(x) \cdot \xi^i \xi^j \leq \Lambda(x) \cdot |\xi|^2$ $\forall \xi \in \mathbb{R}^n \setminus \{0\}$,
where $\lambda(x) =$ min. eigenvalue of $a^{ij}(x)$
$\Lambda(x) =$ max ——, ——
(II) L is strictly elliptic in Ω if $\exists \lambda_0 > 0$ s.t. $\lambda(x) > \lambda_0$ $\forall x \in \Omega$.
(III) L is uniformly elliptic in Ω if L is elliptic Λ/λ is uniformly bounded in Ω.
Rmk: Uniformly elliptic $\not\Rightarrow$ strictly elliptic
Example: Minimal Surface eq.

$$\nabla \cdot \left(\frac{Du}{\sqrt{1+|Du|^2}} \right) = 0$$

i.e. $a^{ij} = \left(\delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2} \right) \cdot \frac{1}{\sqrt{1+|Du|^2}}$

is elliptic, but not uniformly

LECTURE 5

Goal: general 2nd order elliptic operators with $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$: existence & regularity.
No divergence form for L ($a^{ij} \in C^1$).

§2.1 Basic Properties

Theorem 2.2 (Weak Maximum Principle)
Suppose that L is elliptic and that $\sup_\Omega \left|\frac{b^i}{\lambda}\right| < \infty$. Suppose Ω is bounded, open and $u \in C^2(\Omega) \cap C^0(\bar\Omega)$ satisfies $Lu \geq 0$ (i.e. u is a subsolution)
Then

(1) if $c = 0$, then $\sup_\Omega u = \sup_{\partial\Omega} u$

(2) if $c \leq 0$, then $\sup_\Omega u \leq \sup_{\partial\Omega} u^+$
where $u^+ = \max(u, 0)$.

Remark: the assumption that $c \leq 0$ in Ω is crucial: e.g. • $n = 1$, $\Omega = (0, \pi)$, $u'' + u = 0$.
$u(x) = \sin(x)$, with $c \equiv 1$, & $\sup_\Omega u = 1$, $\sup_{\partial\Omega} u^+ = 0$.
• $n = 1$, $\Omega = (0, \pi)^2$, $\Delta u + 2u = 0$
$u(x, y) = \sin(x) \cdot \sin(y)$. Here also, $u|_{\partial\Omega} \equiv 0$

Proof: (1) ($c = 0$). If $Lu > 0$ in Ω, then in fact, SMP holds. Indeed, if $x_0 \in \Omega$ is a local max, then $\partial_i u(x_0) = 0$ & $\partial_i\partial_j u(x_0) \preceq 0$. Since $a^{ij}(x_0) \succ 0$, have

$$a^{ij}\partial_i\partial_j u(x_0) = \mathrm{Tr}\,(A \cdot \nabla^2 u(x_0)) \leq 0.$$

[Briefly: diagonalise both to see that
$$= \mathrm{Tr}\,(\underbrace{\Lambda}_{\geq 0} \cdot \underbrace{(\quad)}_{\leq 0}) \leq 0$$
]

Hence $0 < Lu(x_0) = \underbrace{a^{ij}\partial_i\partial_j u(x_0)}_{\leq 0} + b^i \underbrace{\partial_i u(x_0)}_{= 0}$

$\Rightarrow$ ↯

More generally, if $Lu \geq 0$, consider $v(x) = e^{\gamma x_1}$, $\gamma > 0$ to be chosen.
↳ for any index for which $\sup_\Omega \left|\frac{b^i}{\lambda}\right| < \infty$)

Have $\partial_1 v = \gamma e^{\gamma x_1}$, $\partial_i v = 0$ $\forall i \neq 1$.
$\partial_1\partial_1 v = \gamma^2 e^{\gamma x_1}$, $\partial_i\partial_j v = 0$ $\forall (i,j) \neq (1,1)$.

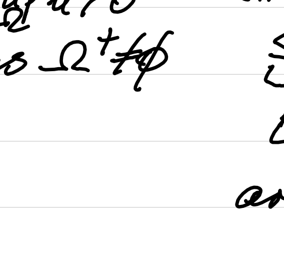
Then
$$Lv = e^{\gamma x_1}(a^{11}\gamma^2 + b^1\gamma) \geq e^{\gamma x_1}(\lambda\gamma^2 + b^1\gamma)$$
$$= \lambda e^{\gamma x_1}\left(\gamma^2 + \frac{b^1}{\lambda}\right) > 0 \text{ in } \Omega \text{ by choosing}$$
γ large.
Since $Lu \geq 0 \Rightarrow L(u + \varepsilon v) > 0$ $\forall \varepsilon > 0$.
Applying the first case, have
$$u(x) \underset{\uparrow v \geq 0}{\leq} \sup(u + \varepsilon v) \leq \sup_{\partial\Omega}(u + \varepsilon v)$$
$$\leq \sup_{\partial\Omega} u + \varepsilon \sup_{\partial\Omega} v$$

Take $\varepsilon \to 0$ to get $u(x) \leq \sup_{\partial\Omega} u$. True $\forall x$, so $\sup_\Omega u \leq \sup_{\partial\Omega} u$. The inequality $\sup_{\partial\Omega} u \leq \sup_\Omega u$ is trivial.

2) ($C \leq 0$). Define
$$L_0 u = a^{ij}\partial_i\partial_j u + b^i \partial_i u$$
& consider $\Omega^+ = \{x \in \Omega : u(x) > 0\}$
Since $cu|_{\Omega^+} \leq 0$

$$L_0 = Lu - cu \geq 0 \text{ on } \Omega^+$$

Note: if $\Omega^+ = \emptyset$, then $u \leq 0$ on Ω and so $u^+ \equiv 0$, so conclusion is trivial. WLOG, assume $\Omega^+ \neq \emptyset$. Then $\partial\Omega^+ \cap \partial\Omega \neq \emptyset$ and $\exists\, x_0 \in \partial\Omega^+ \cap \partial\Omega$ s.t. $u(x_0) \geq 0$.
(If not, then $\partial\Omega^+ \cap \partial\Omega = \emptyset$, then $\partial\Omega^+ \subset \Omega^\circ$



Ω, so $\partial\Omega^+ \subseteq \Omega \setminus \Omega^+$, so $u|_{\partial\Omega^+} \leq 0$. But this contradicts (1) for L_0 in Ω^+.
Hence
$$\sup_\Omega u \underset{\substack{\uparrow \sup_\Omega u > 0 \\ \text{as } \Omega^+ \neq \emptyset}}{=} \sup_{\Omega^+} u \underset{\uparrow (1)}{=} \sup_{\partial\Omega^+} u$$
$$\leq \sup_{\partial\Omega} u \leq \sup_{\partial\Omega} u^+$$
↳ as points on $\partial\Omega^+$ are either in Ω° or $\partial\Omega$

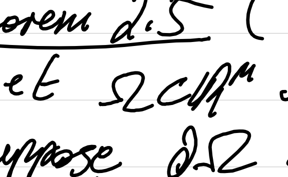
□

Some corollaries
Corollary 2.3 Let Ω be bounded, open $u \in C^2(\Omega) \cap C^0(\bar\Omega)$. Suppose L is elliptic and $\sup_\Omega \left|\frac{b}{\lambda}\right| < \infty$, $c \leq 0$ in Ω.

Then (1) if $Lu \leq 0$ in Ω, then $\inf_\Omega u \geq \inf_{\partial\Omega} u^-$
($u^- = \min(u, 0)$)
(2) if $Lu = 0$, then $\sup_{\partial\Omega} |u| = \sup_\Omega |u|$.

Pf: Exercise
Corollary 2.4: L as above, suppose $u, v, w \in C^2(\Omega) \cap C^0(\bar\Omega)$ satisfy $Lu \geq 0$, $Lv = 0$, $Lw \leq 0$.
Then (i) if $u \leq v$ on $\partial\Omega$, then $u \leq v$ on $\bar\Omega$
(ii) if $v \leq w$ on $\partial\Omega$, then $v \leq w$ on $\bar\Omega$
Pf: Exercise

Want to build towards a SMP.
Theorem 2.5 (Hopf Boundary Point Lemma).
Let $\Omega \subset \mathbb{R}^n$ be open, $y \in \partial\Omega$ and suppose $\partial\Omega$ satisfies the interior sphere condition at y: $\exists R > 0\ \exists z \in \Omega$ s.t. $B_R(z) \subset \Omega$ & $y \in \partial B_R(z)$. Let L be

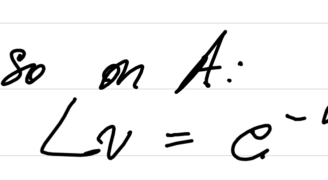


uniformly elliptic in Ω with
$$\sup_\Omega \frac{|b|}{\lambda} + \sup_\Omega \frac{|c|}{\lambda} < \infty$$

Suppose $u \in C^2(\Omega) \cap C^0(\{y\} \cup \Omega)$ and satisfies $u(y) > u(x)$ $\forall x \in \Omega$ & $Lu \geq 0$ in Ω.
Finally, assume one of the following:
(i) $C \equiv 0$ in Ω
(ii) $C \leq 0$ in Ω & $u(y) \geq 0$.
(iii) $u(y) = 0$ (& no assumption on c).

Then $\frac{\partial u}{\partial \nu}(y) > 0$ if it exists, where ν is the outward unit normal at y to $\partial B_R(z)$.
Rmk: $\frac{\partial u}{\partial \nu}(y) \geq 0$ is given by the WMP.

Proof: Let $A = B_R(z) \setminus B_r(z)$ for some $0 < r < R$.



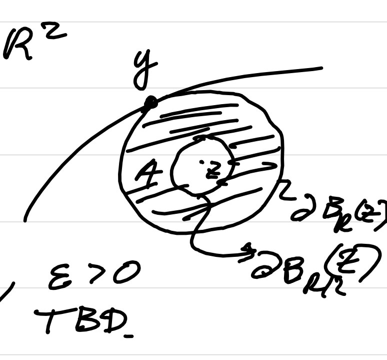
Cases (i) & (ii): on A consider
$$v(x) = e^{-\alpha|x-z|^2} - e^{-\alpha R^2}$$
$$\partial_i v(x) = -2\alpha(x_i - z_i)e^{-\alpha|x-z|^2}$$
$$\partial_i\partial_j v(x) = -2\alpha\,\delta_{ij}\,e^{-\alpha|x-z|^2}$$
$$+ 4\alpha^2(x_i - z_i)(x_j - z_j) \cdot e^{-\alpha|x-z|^2}$$
so on A:
$$Lv = e^{-\alpha|x-z|^2}\big(a_{ij}4\alpha^2(x_i - z_i)(x_j - z_j) - 2\alpha a^{ii} - 2\alpha b^i(x_i - z_i)\big)$$
$$- ce^{-\alpha R^2}$$
ellipticity, $c \leq 0$
$$\geq e^{-\alpha|x-z|^2}\big(4\alpha^2 \lambda(x) \cdot \underbrace{|x-z|^2}_{\geq (R/2)^2 \text{ on } A\ (r > R/2)} - 2\alpha n \Lambda(x)$$
$$- 2\alpha|b| \cdot |x - z| - |c|\big)$$
$$\geq e^{-\alpha|x-z|^2} \lambda(x)\left(\alpha^2 R^2 - 2\alpha n \cdot \sup_\Omega \frac{\Lambda}{\lambda}\right.$$
$$\left.- \alpha R \sup_\Omega \frac{|b|}{\lambda} - \sup_\Omega \frac{|c|}{\lambda}\right)$$

LECTURE 6

Examples Class 1: hand in deadline 2-days before. Hand in "DPMMS pigeonhole "L" or online (Moodle).

Hopf ∂ Point Lemma

Proof: (Cont'd)
constructed
$$v(x) = e^{-\alpha|x-z|^2} - e^{-\alpha R^2}$$
s.t. $Lv \geq 0$ on A.



Put $w(x) = u(x) - u(y) + \varepsilon v(x)$, $\varepsilon > 0$ TBD.

Have $Lw = Lu + \varepsilon Lv - cu(y) \geq 0$ in A by above. Also $v|_{\partial B_R(z)} = 0$ &

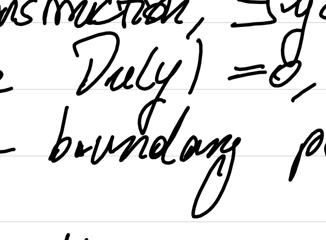
$u(x) \leq u(y)$ on $\overline{\Omega}$, so $w|_{\partial B_R(z)} \leq 0$.
Also, $u(x) < u(y)$ on $\partial B_{R/2}(z)$, so we can choose $\varepsilon > 0$ small enough s.t.
$w|_{\partial B_{R/2}(z)} < 0$. Hence $w|_{\partial A} \leq 0$. Applying WMP to w in A, get
$$u(x) - u(y) + \varepsilon v(x) \leq 0 \quad \text{in } A$$
Choose $t < 0$ so that
$$\frac{u(y+t\nu) - u(y)}{t} \geq -\varepsilon \frac{v(y+t\nu) - \overset{=0}{v(y)}}{t}$$
Sending $t \uparrow 0$: $\frac{\partial u}{\partial \nu}(y) \geq -\varepsilon \frac{\partial v}{\partial \nu}(y) = -\varepsilon \partial_i v(y) \cdot \left(\frac{y_i - z_i}{R}\right)$
$$= 2\alpha\varepsilon R e^{-\alpha R^2} > 0.$$

Case (II) $(u(y) = 0)$
consider $\tilde{L} = L - c^+$ s.t.
$$\tilde{L}u = a^{ij}\partial_i\partial_j u + b^i\partial_i u + \underbrace{(c - c^+)}_{\tilde{c} \leq 0} u$$
Then $\tilde{L}u = \underbrace{Lu}_{\geq 0} - \underbrace{c^+ u}_{\leq c^+ u(y) = 0} \geq 0$
so apply the previous case to $\tilde{L}$ □

Theorem 2.6 (Strong Maximum Principle (SMP)).
Suppose $\Omega \subset \mathbb{R}^n$ is a domain (not necessarily bounded) s.t. $\partial\Omega \neq \phi$ satisfies the interior sphere condition $\forall y \in \partial\Omega$. Let L be uniformly elliptic, $\sup_\Omega \left(\frac{|b|}{\lambda} + \frac{|c|}{\lambda}\right) < \infty$, $u \in C^2(\Omega)$,

$M = \sup_\Omega u < \infty$ and $Lu \geq 0$ in Ω.
Then (i) if $c = 0$ & $u(y) = M$ for some $y \in \Omega$, then $u = M$ in Ω.
(ii) if $c \leq 0$, $M \geq 0$ and $u(y) = M$ for some $y \in \Omega$, then $u \equiv M$ in Ω
(iii) if $M = 0$, & $u(y) = M = 0$ for some $y \in \Omega$, then $u \equiv 0$ in Ω.

Proof: let $\Sigma = \{x \in \Omega : u(x) = M\}$
By continuity, Σ is closed in Ω. Suppose $\Omega \setminus \Sigma \neq \phi$. Pick $z \in \Omega \setminus \Sigma$ s.t. $\text{dist}(z, \partial\Omega) > \text{dist}(z, \partial\Sigma)$.



How?
- first pick $z_1 \in \partial\Sigma \cap \Omega$
- then pick $\rho_1 > 0$ s.t. $B_{\rho_1}(z_1) \subset \Omega$

then pick any $z \in B_{\rho_1/2}(z_1) \setminus \Sigma$.

Then let $R = \sup\{\rho : B_\rho(z) \subset \Omega \setminus \Sigma\}$
By construction, $\exists y \in \partial B_R(z) \cap \Sigma$.
Since $Du(y) = 0$, this contradicts the Hopf-boundary point lemma. So $\Omega \setminus \Sigma = \phi$ $\Rightarrow \Omega = \Sigma$.
The three cases follow directly from Hopf.

Some corollaries:
Corollary 2.7: (Comparison Principle)
Let $L = a^{ij}\partial_i\partial_j + b^i\partial_i + c$ be uniformly elliptic in $\Omega \subset \mathbb{R}^n$ with $\sup_\Omega \left(\frac{|b|}{\lambda} + \frac{|c|}{\lambda}\right) < \infty$. Suppose $u, v \in C^2(\Omega)$ satisfy: $Lu \geq Lv$ & $u \leq v$ in Ω.
Then $u = v$ on Ω or $u < v$ on Ω.
Proof: Have $L(u - v) \geq 0$ in Ω and $u - v \leq 0$ in Ω. If $\exists x_0 \in \Omega$ s.t. $u(x_0) = v(x_0)$, then
$$\text{SMP (ii)} \Rightarrow u \equiv v \text{ in } \Omega.$$
If not, then $u \neq v$ in Ω & so $u < v$ in Ω.

Corollary 2.8: (Uniqueness for Neumann-Problem).
Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain and $\partial\Omega$ satisfies the interior sphere condition at each point. Suppose L is uniformly elliptic with $\frac{|b|}{\lambda} + \frac{|c|}{\lambda} \in L^\infty(\Omega)$ & $c \leq 0$:

Then if $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ s.t.
$$\begin{cases} Lu_i = f & \text{in } \Omega \\ \frac{\partial u_i}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases}$$
for some $f : \Omega \to \mathbb{R}$, $g : \partial\Omega \to \mathbb{R}$, then $u_1 = u_2 + c$, for some constant c.
Proof: $u = u_1 - u_2$ satisfies
$$\begin{cases} Lu = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

Let $M = \sup_\Omega u \geq 0$ (o/w take $-u$)

By SMP if $u \not\equiv M$ on $\overline{\Omega}$, then $\exists y \in \partial\Omega$ s.t. $u(y) = M$ & $u(x) < u(y)$ $\forall x \in \Omega$.
By Hopf ∂ Lemma, $\frac{\partial u}{\partial \nu}(y) \neq 0$, which is a contradiction.
Remark: This says that the trivial Neuman problem with zero data has solutions which are constants, $L(M) = 0$.
But $L(M) = Mc(x)$ $\forall x \in \Omega$,
so if $c \neq 0$, $M = 0$. This constant only non-zero when $c \equiv 0$.

What happens for non-zero RHS?
The following will be critical for Schauder theory.

Theorem 2.9: (Maximum Principle A Priori Estimate (MPAPE))
Suppose $\Omega \subset \mathbb{R}^n$ bounded domain, L elliptic, $c \leq 0$, $\beta = \frac{|b|}{\lambda} \in L^\infty(\Omega)$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$

and $f : \Omega \to \mathbb{R}$.
Then if $Lu \geq f$, then
(i) $\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C \cdot \sup_\Omega \left(\frac{|f|}{\lambda}\right)$
(ii) if $Lu = f$, then
$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + C \sup_\Omega \left(\frac{|f|}{\lambda}\right)$$
$C = C(\beta, \text{diam}(\Omega))$.
Proof: Put $d = \text{diam}(\Omega) = \sup_{x,y \in \Omega} |x - y|$

As Ω is bounded, have
$$\Omega \subset \{x : a \leq x_1 \leq a + d\}$$
for some $a \in \mathbb{R}$, WLOG $a = 0$.

Idea: construct subsolution & use WMP
Let $v(x) = \sup_{\partial\Omega} u^+ + (e^{\alpha d} - e^{\alpha x_1}) \sup_\Omega \frac{|f|}{\lambda}$

where α TBD.
Compute:
$$(a^{ij}\partial_i\partial_j + b^i\partial_i)e^{\alpha x_1} = e^{\alpha x_1}(a^{11}\alpha^2 + b^1\alpha)$$
by ellipticity ($a^{11} \geq \lambda$)
$$\geq e^{\alpha x_1}\lambda\left(\alpha^2 + \frac{b^1}{\lambda}\alpha\right)$$
$$\geq e^{\alpha x_1}\lambda(\alpha^2 - \beta\alpha)$$
$(\alpha = \beta + 1)$ $\geq \lambda$.
Hence
$$Lv = (a^{ij}\partial_i\partial_j + b^i\partial_i)\left(-e^{\alpha x_1}\sup \frac{|f|}{\lambda}\right) + cv$$
$$\leq cv - \lambda \sup \frac{|f|}{\lambda}$$
$$\leq -\lambda \sup_\Omega \frac{|f|}{\lambda}$$

Then (i) if $Lu \geq f$, then $L(u - v) \geq f + \lambda \sup \frac{|f|}{\lambda}$
$$= \lambda\left(\frac{f}{\lambda} + \sup_\Omega \frac{|f|}{\lambda}\right) \geq 0.$$

To be Continued...

LECTURE 7

Maximum Principle A Priori Estimate

Proof: (continued)

Had $v(x) = \sup_{\partial\Omega} u^+ + \left(e^{(\beta+1)d} - e^{(\beta+1)x_1}\right) \times \sup_{\Omega} \frac{|f|}{\lambda}$

& showed $L(u-v) \geq 0$.

Since $u \leq u^+$, $u|_{\partial\Omega} \leq v|_{\partial\Omega}$ (from def^n of v). So by the WMP, $u \leq v$ in Ω, so $\sup_{\Omega} u \leq \sup_{\Omega} v \leq \sup_{\partial\Omega} u^+ + C \cdot \sup_{\Omega} \frac{|f|}{\lambda}$

where $C = \sup_{\Omega}\left(e^{(\beta+1)d} - e^{(\beta+1)x_1}\right)$

(II) $Lu = f$, then apply (I) to $-u$ and combine □

§2.2 Hölder Spaces

Fix $\Omega \subset \mathbb{R}^n$ open, let $\alpha \in (0,1]$.

Definition 2.10: We say that $u: \Omega \to \mathbb{R}$ is uniformly Hölder continuous with exponent α or uniformly α-Hölder continuous if

$$[u]_{\alpha,\Omega} := \sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} < \infty$$

This is the Hölder semi-norm.
If $\alpha = 1$, this says u is uniformly Lipschitz. If $\alpha > 1$, that would make $u = \text{const}$ (MVT).

Definition 2.11: We say that u is locally α-Hölder continuous in Ω if $\forall K \subset\subset \Omega$, $u|_K : K \to \mathbb{R}$ is uniformly α-Hölder continuous.

Let $k \in \mathbb{N} \cup \{+\infty\}$. Recall for a multi-index $\beta \in \mathbb{N}^n$, $|\beta| = \sum_i \beta_i$.
and $C^k(\Omega) = \{u : \Omega \to \mathbb{R} : D^\beta u \text{ exists and is continuous } \forall \beta : |\beta| \leq k\}$

Definition 2.12: define the Hölder spaces $C^{k,\alpha}(\Omega)$:

$$C^{k,\alpha}(\Omega) = \{u \in C^k(\Omega) : D^\beta u \text{ is locally } \alpha\text{-Hölder cont. } \forall \beta \text{ s.t. } |\beta| = k\}$$

& $C^{k,\alpha}(\overline{\Omega}) = \{ \text{ — '' — uniformly } \alpha\text{-Hölder cts} \}$.

We write for $\alpha \in (0,1)$
$C^\alpha(\Omega) = C^{0,\alpha}(\Omega)$
$C^\alpha(\overline{\Omega}) = C^{0,\alpha}(\overline{\Omega})$
$C^{k,0}(\Omega) := C^k(\Omega)$,
$C^{k,0}(\overline{\Omega}) := C^k(\overline{\Omega})$, $k \in \mathbb{N} \cup \{+\infty\}$.

Remark: Note $C^{k+1}(\Omega) \neq C^{k,1}(\Omega)$, indeed:
Lipschitz $\not\Rightarrow$ ctsly diff-able.
(But Lipschitz $\Rightarrow$ diff-able a.e.)

Finally, define $C_0^{k,\alpha}(\Omega) \equiv C_c^{k,\alpha}(\Omega) := \{u \in C^{k,\alpha}(\Omega) : \text{supp}(u) \subset \Omega \text{ is compact}\}$

$(\text{supp}(u) = \overline{\{x \in \Omega : u(x) \neq 0\}})$.

To get norms on these spaces, put for $k \in \mathbb{N}$, $u \in C^k(\overline{\Omega})$

$$[u]_{k,\Omega} \equiv |D^k u|_{0,\Omega} = \sup_{|\beta|=k} |D^\beta u|_{0,\Omega} = \sup_{|\beta|=k}\left(\sup_{x\in\Omega} |D^\beta u(x)|\right)$$

recall for $\alpha = 0$ norm is C^0 norm

For $u \in C^{k,\alpha}(\overline{\Omega})$, put

$$[u]_{k,\alpha,\Omega} \equiv [D^k u]_{\alpha,\Omega} := \sup_{|\beta|=k} [D^\beta u]_{\alpha,\Omega}$$

Note these are semi-norms. To get norms,

$$\|u\|_{C^k(\overline{\Omega})} \equiv |u|_{k,\Omega} \equiv |u|_{k,0,\Omega} \equiv \sum_{j=0}^{k} |D^j u|_{0,\Omega} \text{ and}$$

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} \equiv |u|_{k,\alpha,\Omega} := |u|_{k,\Omega} + [D^k u]_{\alpha,\Omega}.$$

With these norms, C^k & $C^{k,\alpha}$ become Banach spaces.

Important to understand Compactness Properties.

Theorem 2.13 (Arzelà-Ascoli for Hölder Spaces). Let $\Omega \subset \mathbb{R}^n$ open, $k \in \mathbb{N}$, $\alpha \in (0,1]$. If $(u_j)_j \subset C^{k,\alpha}(\Omega)$ satisfies

$$\sup_j |u_j|_{k,\alpha,\Omega'} < \infty$$

$\forall \Omega' \subset\subset \Omega$; then $\exists u \in C^{k,\alpha}(\Omega)$ and a subsequence $(u_{j'})_{j'}$ s.t. $u_{j'} \to u$ in $C^k(\overline{\Omega'})$

Remark: Nothing is said about convergence in $C^{k,\alpha}(\overline{\Omega'})$.

Proof: (ES2) □

Two more ingredients before starting Schauder theory.

Ingredient 1:

Interpolation: If we have Banach spaces $X \subset\subset Y \subset Z$, then we can bound the norm in Y by X & Z norms.
Interpolation is the exchange of sizes of X- & Z-norms.
Here $C^{k,\alpha}(\overline{\Omega}) \subset\subset C^k(\overline{\Omega}) \subset C^0(\overline{\Omega})$

Theorem 2.14 (Interpolation Inequality for Hölder Spaces). Let $\varepsilon > 0$, $l \in \mathbb{N}$, $\alpha \in (0,1]$.
Then $\exists C = C(n,l,\alpha,\varepsilon) \in (0,\infty)$ s.t.
if $u \in C^{l,\alpha}(\overline{B_R(x_0)})$, then

$$R^k \cdot |D^k u|_{0,B_R(x_0)} \leq \varepsilon \cdot R^{l+\alpha} [D^l u]_{\alpha;B_R} + C \cdot |u|_{0,B_R(x_0)}.$$

$\forall\ 0 \leq k \leq l$.

Sketch Proof: (Details in ES2)
By rescaling and shifting, i.e. considering $v(x) = u(x_0 + Rx)$ suffices to prove for $R = 1$, $x_0 = 0$. Then argue by contradiction and Arzelà-Ascoli.

Ingredient 2.15: (Simon's Absorbing Lemma)
Let $B_R(x) \subset \mathbb{R}^n$ be fixed & S a non-negative, sub-additive function on the collection of sub-balls of $B_R(x)$ i.e. if

$$B_\rho(y) \subset \bigcup_{j=1}^{N} B_{\rho_j}(y_j) \subset B_R(x)$$

Then $S(B_\rho(y)) \leq \sum_{j=1}^{N} S(B_{\rho_j}(y_j))$

Let $\lambda \in [0,\infty)$, $\theta \in (0,1]$. Then $\exists\ \delta = \delta(n,\lambda,\theta) \in (0,1)$ s.t. the following holds:
Suppose that for all balls $B_\rho(y) \subset B_R(x)$ we have:

$$\rho^\lambda S(B_{\theta\rho}(y)) \leq \delta \rho^\lambda S(B_\rho(y)) + \gamma$$

for some fixed $\gamma > 0$. Then

$$R^\lambda S(B_{\theta R}(x)) \leq C\gamma$$

for some $C = C(n,\theta,\lambda)$.

Remark: This says that if $\exists$ a local bound on S, then we can "absorb" the S-term on the RHS to get a global bound.

Next time: proof.

LECTURE 8

Proof of Simon's Absorbing Lemma:

Put $Q = \sup_{B_\rho(y) \subset B_R(x)} \rho^\lambda S(B_{\theta\rho}(y))$

Recall we had

$(*)$ $\rho^\lambda S(B_{\theta\rho}(y)) \leq \delta \cdot \rho^\lambda S(B_\rho(y)) + \gamma.$

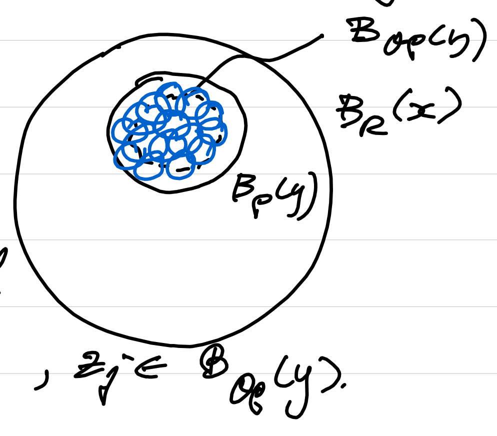
By the subadditivity of S, have

$$Q \leq R^\lambda S(B_R(x)) < \infty.$$

Fix any $B_\rho(y) \subset B_R(x)$. Cover $B_{\theta\rho}(y)$ by a collection of balls

$\{B_{(1-\theta)\theta^2\rho}(z_j)\}_{j=1}^N$, $N \leq c(\theta, n)$ indep of ρ & y.

$z_j \in B_{\theta\rho}(y)$.



How? Chose a maximal, pairwise disjoint collection of balls $\{B_{\frac{(1-\theta)\theta^2\rho}{2}}(z_j)\}_{j=1}^N$, $z_j \in B_{\theta\rho}(y)$.

We claim that these z_j's work. Indeed, if not, then $\exists z \in B_{\theta\rho}(y) \setminus \bigcup_{j=1}^N B_{(1-\theta)\theta^2\rho}(z_j)$ & we have that $d(z, z_j) \geq (1-\theta)\theta^2\rho \ \forall j$.

So $B_{\frac{(1-\theta)\theta^2\rho}{2}}(z) \cap B_{\frac{(1-\theta)\theta^2\rho}{2}}(z_j) = \emptyset$.

which contradicts maximality.

Bound on N:

Note (from considering the radii)

$$\bigcup_{j=1}^N B_{\frac{(1-\theta)\theta^2\rho}{2}}(z_j) \subset B_{\underbrace{\frac{(1-\theta)\theta^2\rho}{2} + \theta\rho}_{<\rho}}(y).$$

Since the LHS is disjoint, $\exists$ volume bound.

$$N\omega_n \left(\frac{(1-\theta)\theta^2\rho}{2}\right)^n \leq \omega_n \left(\frac{(1-\theta)\theta^2\rho}{2} + \theta\rho\right)^n$$

$$\Rightarrow N \leq \frac{\left(\frac{(1-\theta)\theta}{2} + 1\right)^n}{\left(\frac{(1-\theta)\theta}{2}\right)^n}$$ indep. of ρ & y.

By sub-additivity, we have

$$\rho^\lambda S(B_{\theta\rho}(y)) \leq \rho^\lambda \sum_{j=1}^N S(B_{(1-\theta)\theta\rho}(z_j))$$

$$(*)\ \rho \to (1-\theta)\theta\rho \Rightarrow \leq ((1-\theta)\theta)^{-\lambda} \sum_{j=1}^N \Big(\underbrace{\delta((1-\theta)\theta\rho)^\lambda S(B_{(1-\theta)\theta\rho}(z_j))}_{\leq Q} + \gamma \Big)$$

$$\leq \delta((1-\theta)\theta)^{-\lambda} N Q + N\gamma((1-\theta)\theta)^{-\lambda}$$

Now take the sup over all $B_\rho(y) \subset B_R(x)$:

$$Q \leq \delta C_1 Q + C_2 \gamma$$

where C_1, C_2 depend on n, θ, λ. Then choose $\delta > 0$ small enough ($= 1/2C_1$), then $Q \leq 2C_2 \cdot \gamma$.

§3 Schauder Theory:

§3.1 Interior Schauder Estimates:

We will first prove interior estimates in the unit ball, and then extend them to more general domains.

Main point: if coefficients of L are α-Hölder cts, then any $C^{2,\alpha}$ solution of $Lu = f$ can be bounded in $C^{2,\alpha}$ on a smaller ball by $|u|_0$ & f.

Theorem 3.1: (Unit Scale Interior Schauder Estimates). Let $\alpha \in (0,1)$, $\beta > 0$, & suppose $a^{ij}, b^i, c \in C^{0,\alpha}(B_1(0))$ with

$$|a^{ij}|_{0,\alpha; B_1(0)} + |b^i|_{0,\alpha; B_1(0)} + |c|_{0,\alpha; B_1(0)} \leq \beta$$

Suppose L strictly elliptic, i.e. $\exists \lambda > 0$ s.t.

$$a^{ij}\xi^i\xi^j \geq \lambda|\xi|^2 \quad \forall x \in B_1(0), \xi \in \mathbb{R}^n.$$

Then if $u \in C^{2,\alpha}(B_1(0)) \cap C^{0,\alpha}(\overline{B_1(0)})$ and $f \in C^{0,\alpha}(B_1(0))$ satisfy $Lu = f$ in $B_1(0)$, then

$$|u|_{2,\alpha; B_{\frac{1}{2}}(0)} \leq C \cdot (|u|_{0; B_1(0)} + |f|_{0,\alpha; B_1(0)})$$

for some constant $C = C(n, \lambda, \alpha, \beta)$.

Remarks:

- can never take $\alpha = 0$, or $\alpha = 1$ in these cases (Thm 3.1 $|\alpha = 0, \alpha = 1$ is false! ES1).
- strict ellipticity gives lower bound for λ and upper bound on $|a^{ij}|_{0,\alpha; B_1(0)}$ gives an upper bound on Λ. So Λ is bounded above so have strict λ ellipticity $\Rightarrow$ uniform ellipticity.
- Remarkable result: $\sup_{B_1(0)} |u|$ & f control two derivatives of u!
- Will in fact strengthen this to
  $|u|_{2,\alpha; B_\theta(0)} \leq C(|u|_{0, B_1(0)} + |f|_{0,\alpha; B_1(0)})$
  $\forall \theta \in (0,1)$, $C = c(n, \lambda, \alpha, \beta, \theta)$.
- There are no assumptions or conclusions about the C^2-norm up to the boundary.
- The Schauder estimate gives a compactness property for the space of solutions to $Lu = f$. If $(u_k)_k \subset C^{2,\alpha}(B_1(0)) \cap C^0(\overline{B_1(0)})$ solve $Lu_k = f$ in $B_1(0)$ and
  $$\gamma = \sup_k \sup_{B_1(0)} |u_k| < \infty$$

Then estimate $\Rightarrow$ $|u_k|_{2,\alpha; B_\theta(0)} \leq C(\gamma, n, \theta, \beta, \alpha, \lambda, f)$.

So by Arzela-Ascoli, $\exists$ subsequence $(u_{k'})_{k'}$ & $u \in C^{2,\alpha}(B_1(0))$ s.t. $u_{k'} \to u$ in $C^2(B_\theta(0))$ $\forall \theta \in (0,1)$. Passing to the limit, then $Lu = f$.

Proof. Write $B_r := B_r(0)$. Working in a slightly smaller ball, we can assume WLOG that $|u|_{2, B_1} < \infty$. Three steps

1. reduction step
2. contradiction step
3. simplified PDE step.

Step 1: Reduction Step.

Claim: It suffices to prove the following: for any given $\delta \in (0,1)$, $\exists C > 0$ s.t.

(3.1) $$[D^2u]_{\alpha; B_{1/2}} \leq \delta \cdot [D^2u]_{\alpha; B_1} + C(|u|_{2, B_1} + |f|_{0,\alpha; B_1})$$

Proof of claim:

Suppose $Lu = f \Rightarrow u$ satisfies (3.1)

By Hölder interpolation inequality, then

$$[D^2u]_{\alpha; B_{1/2}} \leq 2 \cdot \delta [D^2u]_{\alpha; B_1} + C \cdot (|u|_{0, B_1} + |f|_{0,\alpha; B_1})$$

Strategy for step 1: take $B_\rho(y) \subset B_1(0)$, & shift and scale: $\tilde{u}(x) = u(y + \rho x)$. Then $\tilde{u}$ will satisfy a new PDE & a new inequality (3.2). To be continued.

LECTURE 9

Take any sub-ball $B_\rho(y) \subset B_1(0)$ & $\tilde{u}(x) = u(y+\rho x)$. Then $\tilde{u}$ satisfies $\tilde{a}^{ij}\partial^2_{ij}\tilde{u} + \tilde{b}^i\partial_i\tilde{u} + \tilde{c}\tilde{u} = \tilde{f}$ where

$$\tilde{a}^{ij}(x) = a^{ij}(y+\rho x)$$
$$\tilde{b}^i(x) = \rho\, b^i(y+\rho x)$$
$$\tilde{c}(x) = \rho^2 c(y+\rho x)$$
$$\tilde{f}(x) = \rho^2 f(y+\rho x).$$

Further

$$|\tilde{a}^{ij}|_{0,\alpha;B_1(0)} + |\tilde{b}^i|_{0,\alpha;B_1(0)} + |\tilde{c}|_{0,\alpha;B_1}$$
$$\leq |a^{ij}|_{0;B_\rho(y)} + \rho^\alpha [a^{ij}]_{\alpha;B_\rho(y)}$$
$$+ \rho |b^i|_{0;B_\rho(y)} + \rho^{1+\alpha}[b^i]_{\alpha;B_\rho(y)}$$
$$+ \text{ similar for } c$$
$$\leq \beta. \qquad \text{as } \rho < 1.$$

Since $\tilde{a}^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$, the PDE $(*)$ is strictly elliptic. So by assumption (3.2) holds for $\tilde{u}$, call it $(\tilde{3.2})$. Expressing $(\tilde{3.2})$ in terms of u gives:

$$\rho^{2+\alpha}[D^2u]_{\alpha;B_{\rho/2}} \leq 2\delta\rho^{2+\alpha}[D^2u]_{\alpha;B_\rho(y)} + C\big(|u|_{0;B_\rho(y)} + \rho^2|f|_{0;B_\rho(y)} + \rho^{2+\alpha}[f]_{\alpha;B_\rho(y)}\big)$$
$$\leq 2\delta\rho^{2+\alpha}[D^2u]_{\alpha;B_\rho(y)} + \underbrace{C\big(|u|_{0;B_1} + |f|_{0,\alpha;B_1}\big)}_{:=\gamma,\ \text{indep. of } \rho \text{ and } y}.$$

So by the absorbing lemma, choose δ similarly, have
$$[D^2u]_{\alpha;B_{1/2}} \leq C\big(|u|_{0;B_1} + |f|_{0,\alpha;B_1}\big)$$
where C depends only on $n, \alpha, \lambda, \beta$. This is the conclusion of the theorem (use interpolation again).

Step 2: Contradiction & Arzela–Ascoli.

Suppose $\exists\,\delta$ s.t. $\forall k \in \mathbb{N}$ $\exists\, a^{ij}_k, b^i_k, c_k$ s.t. $|a^{ij}_k|_{0,\alpha;B_1} + |b^i_k|_{0,\alpha;B_1} + |c_k|_{0,\alpha;B_1} \leq \beta$ (β indep. of k) & $a^{ij}_k\xi_i\xi_j \geq \lambda\cdot|\xi|^2$ & $u_k \in C^{2,\alpha}(B_1) \cap C^{0,\alpha}(\overline{B_1})$ solving $Lu_k = f_k$, for $f_k \in C^{0,\alpha}(B_1)$, but

$$(3.3)\quad [D^2u_k]_{\alpha;B_{1/2}} > \delta[D^2u_k]_{\alpha;B_1} + k\big(|u_k|_{2;B_1} + |f_k|_{0,\alpha;B_1}\big)$$

By definition of $[D^2u_k]_{\alpha;B_{1/2}}$ and by passing to a subsequence, we may assume $\exists\, x_k, y_k \in B_{1/2}$ and fixed l,m s.t.:

$$\frac{|D_{lm}u_k(x_k) - D_{lm}u_k(y_k)|}{|x_k - y_k|^\alpha} \geq \tfrac{1}{2}[D^2u_k]_{\alpha;B_{1/2}}.$$

(by taking an appropriate subsequence in l_k, m_k $\forall k$). Let $\rho_k = |x_k - y_k|$. Then

$$\tfrac{1}{2}[D^2u_k]_{\alpha;B_{1/2}} < \frac{|D_{lm}u_k(x_k)| + |D_{lm}u_k(y_k)|}{\rho_k^\alpha}$$
$$\leq \frac{2|u_k|_{2;B_1}}{\rho_k^\alpha}$$
$$\overset{(3.3)}{\leq} \frac{2[D^2u_k]_{\alpha;B_{1/2}}}{k\rho_k^\alpha}$$

So in particular $\rho_k^\alpha < \frac{4}{k} \to 0$

(note $\alpha = 0$ does not imply $\rho_k \to 0$).

Next rescale appropriately & take the limit, set

$$\tilde{u}_k(x) = \frac{u_k(x_k + \rho_k x) - q_k(x)}{\rho_k^{2+\alpha}[D^2u_k]_{\alpha;B_1}}$$

where $q_k(x) = u_k(x_k) + \rho_k\cdot x^i\partial_i u_k(x_k) + \frac{\rho_k^2}{2}x_i x_j\partial_i\partial_j u_k(x_k)$

Note x has nothing to do with x_k.

By construction $\tilde{u}_k(0) = 0$, $D\tilde{u}_k(0) = 0$, $D^2\tilde{u}_k(0) = 0$, & u_k is defined on B_1

$\Rightarrow \tilde{u}_k$ defined on $B_{1/\rho_k}\left(-\frac{x_k}{\rho_k}\right) \supseteq \underbrace{B_{1/2\rho_k}(0)}_{x_k \in B_{1/2}(0)}$.

So by direct calculation
$$[D^2\tilde{u}_k]_{\alpha;B_{\frac{1}{2\rho_k}}(0)} \leq 1.$$

Taylor with remainder

Hence for any $R \geq 1$ (using (3.3) to control $|\tilde{u}_k|_{2,B_R}$): $|\tilde{u}_k|_{2,\alpha;B_R} \leq C\cdot R^{2+\alpha}$.

Hence by Arzela–Ascoli, passing to a subsequence, $\exists\, v \in C^{2,\alpha}(\mathbb{R}^n)$ s.t. $\tilde{u}_k \to v$ in $C^2(B_R(0))$ $\forall R > 0$ & s.t.

$$[D^2v]_{\alpha;\mathbb{R}^n} \leq 1.$$

Step 3: Find a PDE for v.

First put $w_k(x) = u_k(x_k + \rho_k x)$.

This satisfies $\tilde{L}_k w_k = \tilde{f}_k$ in $B_{\frac{1}{2\rho_k}}(0)$ where

$$\tilde{L}_k = \tilde{a}^{ij}_k\partial^2_{ij} + \tilde{b}^i_k\partial_i + \tilde{c}_k,$$

where $\tilde{a}^{ij}_k$ are as before with $\rho \mapsto \rho_k$ and $x_k \mapsto y$, i.e. $\tilde{a}^{ij}_k = a^{ij}_k(x_k + \rho_k x)$ etc.

Have $w_k(x) = \rho_k^{2+\alpha}[D^2u_k]_{\alpha;B_1}\tilde{u}_k(x) + q_k(x)$

& also $\tilde{L}_k\tilde{u}_k = g_k = \dfrac{\tilde{f}_k - \tilde{L}_k q_k}{\rho_k^{2+\alpha}[D^2u_k]_{\alpha;B_1}}$

will find that $\tilde{L}_k$ tends uniformly to a constant coefficient operator, and $g_k \to$ constant. Indeed,

$$[\tilde{a}^{ij}_k]_{\alpha;B_{\frac{1}{2\rho_k}}} \underset{\text{scaling \& containment}}{\leq} \rho_k^\alpha[a^{ij}_k]_{\alpha;B_1} \leq \rho_k^\alpha\beta \to 0 \text{ as } k \to \infty \quad (\alpha > 0).$$

Remark $\alpha = 0$ fails here!

By Arzela–Ascoli on $\tilde{a}^{ij}_k$:
get $\tilde{a}^{ij}_k \to \tilde{a}^{ij} \in C^{0,\alpha}(\mathbb{R}^n)$ locally uniformly, and the limit has $[\tilde{a}^{ij}]_{\alpha;\mathbb{R}^n} = 0$ $\forall \alpha > 0$.

Hence $\tilde{a}_{ij}$ is constant. Also,

(Check!) $\left.\begin{array}{l} |\tilde{b}^i_k|_{0;B_{\frac{1}{2\rho_k}}} \leq \rho_k\beta \to 0 \\ |\tilde{c}_k|_{0;B_{\frac{1}{2\rho_k}}} \leq \rho_k^2\beta \to 0 \end{array}\right.$

So $\tilde{b}^i_k \to 0$ & $\tilde{c}_k \to 0$ locally uniformly on $\mathbb{R}^n$.

Finally $g_k \to 0$ locally uniformly (Proof: ES2)

Taking the limit in $\tilde{L}_k\tilde{u}_k = g_k$, we obtain:

$$(3.4)\quad \tilde{a}^{ij}D^2_{ij}v = 0 \text{ on } \mathbb{R}^n$$

constants

Also, in the limit $\tilde{a}^{ij}\xi_i\xi_j \geq \lambda\cdot|\xi|^2$, so we are still strictly elliptic. Diagonalise $A = (\tilde{a}^{ij})$,

$$PAP^T = Q = \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$$

$\lambda_i \geq \lambda > 0$ $\forall i$. Let $w(x) = v(P^T x)$

Then $D^2v(x) = P^T D^2w(Px)P$,

so (3.4) becomes $0 = \operatorname{tr}(AD^2v)$, trace cyclic
$$= \operatorname{tr}(QD^2w(Px))$$
$$= \textstyle\sum_{i=1}^n \lambda_i D^2_{ii}w(Px)$$

$\Rightarrow \sum_{i=1}^n \lambda_i\cdot D^2_{ii}w(x) = 0.$

Rescaling, $\tilde{w}(x) = w(\sqrt{\lambda_1}x_1, \dots, \sqrt{\lambda_n}x_n)$

$\Rightarrow \Delta\tilde{w} = 0$ on $\mathbb{R}^n$ &

$[D^2\tilde{w}]_{\alpha;\mathbb{R}^n} < \infty$

$\overset{\text{(Weyl)}}{\Longrightarrow}$ $\tilde{w}$ is smooth & in particular

$$\Delta(D^2_{ij}\tilde{w}) = 0 \text{ on } \mathbb{R}^n$$

But by Hölder continuity,
$$|D^2_{ij}\tilde{w}(x)| \leq |D^2_{ij}\tilde{w}(0)| + [D^2\tilde{w}]_{\alpha;\mathbb{R}^n}|x|^\alpha$$

here cannot use Liouville for $\alpha = 1$

$\Rightarrow$ Liouville's thm $\Rightarrow$ $\underline{\underline{D^2_{ij}\tilde{w} = \text{const.}}}$

LECTURE 10

Recall found $\tilde{w}$ s.t. $D_{ij}v = \text{constant}$ as a limit of $\tilde{u}_k \to v$ in C^2.
But $D^2_{ln}v(0) = 0$ so, $D^2 v \equiv 0$.
On the other hand, consider $\xi_k = \frac{x_k - y_k}{\rho_k}$
$|\xi_k| = 1$, and
$$u_k(x_k + \rho_k \xi_k) = u_k(y_k)$$
So $$|D^2_{ln}\tilde{u}_k(\xi_k)| = \left| \frac{\rho_k^2 \cdot (D^2_{ln}u_k(y_k) - D^2_{ln}u_k(x_k))}{\rho_k^{2+\alpha}[D^2 u_k]_{\alpha;B_1}} \right|$$
by choice of x_k, y_k
$$\geq \frac{1}{2}\frac{[D^2 u_k]_{\alpha;B_{1/2}}}{[D^2 u_k]_{\alpha;B_1}}$$
$$\overset{(3.3)}{>} \delta/2$$
Since ξ_k is bdd and have, up to a subsequence, $\xi_k \to \xi$, then in the limit $|D^2_{ln}v(\xi)| \geq \delta/2$.
This contradicts $D^2 v \equiv 0$ $\square$

So we have proven
$$|u|_{2,\alpha;B_{1/2}} \leq C(|u|_{0;B_1} + |f|_{0,\alpha;B_1}).$$

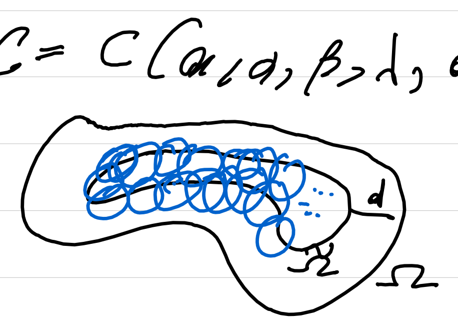
We now give some corollaries.

Corollary 3.2: (Scale-invariant interior Schauder Estimate). Suppose $B_R(x_0) \subset \mathbb{R}^n$ and $a^{ij}, b^i, c \in C^{0,\alpha}(B_R(x_0))$ are strictly elliptic, $a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$, $\lambda > 0$, $\forall x \in B_R(x_0)$ $\forall \xi \in \mathbb{R}^n$. Suppose also that
$$|a^{ij}|_{0;B_R(x_0)} + R^\alpha[a^{ij}]_{\alpha;B_R(x_0)} + R\cdot(|b^i|_{0;B_R(x_0)} + R^\alpha[b^i]_{\alpha;B_R(x_0)}) + R^2\cdot(|c|_{0;B_R(x_0)} + R^\alpha[c]_{\alpha;B_R(x_0)}) \leq \beta.$$
for some $\beta \geq 0$. Suppose $u \in C^{2,\alpha}(B_R(x_0)) \cap C^{0,\alpha}(\overline{B_R(x_0)})$ satisfies $Lu = f \in C^{0,\alpha}(B_R)$.
Then,
$$|u|'_{2,\alpha;B_{\frac{R}{2}}(x_0)} \leq C(|u|_{0;B_R(x_0)} + R^2|f|_{0;B_R(x_0)} + R^{2+\alpha}[f]_{\alpha;B_R(x_0)}).$$
where $|u|'_{k,\alpha;B_\rho(y)} = \sum_{j=0}^{k} \rho^j |D^j u|_{0;B_\rho(y)} + \rho^{k+\alpha}[D^k u]_{\alpha;B_\rho(y)}$
& $C = C(n, \lambda, \alpha, \beta)$ (indep. of u & R).

Proof: Apply Theorem (3.1) with $x \mapsto x_0 + Rx$ $\square$

Corollary 3.3 (Interior Schauder Estimates in General Domains).
Let $\alpha \in (0,1)$ and let $\Omega \subset \mathbb{R}^n$ open, bdd, and suppose that $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$ where
$$|a^{ij}|_{0,\alpha;\Omega} + |b^i|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \leq \beta$$
with $a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$, $\lambda > 0$ $\forall x \in \Omega$ $\forall \xi \in \mathbb{R}^n$.
Suppose $u \in C^{2,\alpha}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ solves $Lu = f \in C^{0,\alpha}(\Omega)$. Then $\forall \tilde{\Omega} \subset\subset \Omega$
$$|u|_{2,\alpha;\tilde{\Omega}} \leq C\cdot(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega})$$
where $C = C(n, \alpha, \beta, \lambda, \operatorname{dist}(\tilde{\Omega}, \partial\Omega))$.

Proof:



Let $d = \operatorname{dist}(\tilde{\Omega}, \partial\Omega) = \sup\{r > 0 : (\tilde{\Omega})_r \subset \Omega\}$.
where $(\tilde{\Omega})_r = \bigcup_{x \in \tilde{\Omega}} B_r(x)$ is the r-neighbourhood of $\tilde{\Omega}$. Then $\forall x \in \tilde{\Omega}$ $B_d(x) \subset \Omega$, so
$$|a^{ij}|'_{0,\alpha;B_d(x)} + d|b^i|'_{0,\alpha;B_d(x)} + d^2|c|'_{0,\alpha;B_d(x)} \leq C(d)\cdot\beta$$
Then by Corollary (3.2), get
$$|u|_{0;B_{d/2}(x)} + d|Du|_{0;B_{d/2}(x)} + d^2|D^2u|_{0;B_{d/2}(x)} + d^{\alpha+2}[D^2u]_{\alpha;B_{d/2}(x)}$$
$$\leq C\cdot(|u|_{0;B_d(x)} + d^2|f|_{0;B_d(x)} + d^{\alpha+2}[f]_{\alpha;B_d})$$
$$\leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}) \qquad \underline{(3.4)}$$
& $C = C(n, \lambda, \alpha, \beta, d)$.

In particular,
$$|u(x)| + |Du(x)| + |D^2u(x)| \leq C\cdot\underbrace{(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega})}_{RHS}$$
$\forall x \in \tilde{\Omega}$. So
$$(*) \quad |u|_{2;\tilde{\Omega}} \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega})$$
But also by (3.4)
$$\sup_{\substack{x,y \in \tilde{\Omega} \\ |x-y| < d/2}} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \leq C\cdot RHS.$$

On the other hand, if $|x-y| \geq d/2$, then
$$\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \leq \left(\frac{d}{2}\right)^{-\alpha}\cdot 2\cdot|u|_{2;\tilde{\Omega}}$$
Hence, $[D^2u]_{\alpha;\tilde{\Omega}} \leq C\cdot RHS$ $\qquad (**)$
& so combine $(*)$ & $(**)$ to conclude. $\square$

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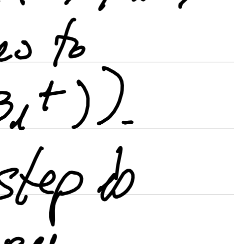
§ 3.2: Boundary Schauder Estimates

Write $\mathbb{R}^n_\pm = \{(x', x^n) : x' \in \mathbb{R}^n, x^n \gtrless 0\}$
$B^\pm_R(y) = B_R(y) \cap \mathbb{R}^n_\pm$
$B^\pm_R := B^\pm_R(0)$
$S_R(y) = B_R(y) \cap \{x_n = 0\}$
$S_R = S_R(0)$.



S_R

Theorem 3.4: (Boundary Schauder Estimates a Unit Ball). As before $0 < \alpha < 1$,
$a^{ij}, b^i, c \in C^{0,\alpha}(B_1^+)$, &
$$|a^{ij}|_{0,\alpha;B_1^+} + |b^i|_{0,\alpha;B_1^+} + |c|_{0,\alpha;B_1^+} \leq \beta$$
& $a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$, $\forall x \in B_1^+$ $\forall \xi \in \mathbb{R}^n$.
Suppose $u \in C^{2,\alpha}(B_1^+)$ solves:
$$\begin{cases} Lu = f \in C^{0,\alpha} & \text{in } B_1^+ \\ u = \varphi \in C^{2,\alpha}(B_1^+) & \text{on } S_1 \end{cases}$$
Then $|u|_{2,\alpha;B^+_{1/2}} \leq C(|u|_{0,B_1^+} + |f|_{0,\alpha;B_1^+} + |\varphi|_{2,\alpha;B_1^+})$

Proof: By considering $v := u - \varphi$, suffices to consider the case $\varphi \equiv 0$ ($\varphi \in C^{2,\alpha}(B_1^+)$).
Proceed as in Thm 3.1. Reduction step is exactly the same. Steps 2 & 3 are key.
↳ Absorbtion lemma still holds, same counting)

Step 2:
Claim: $\forall \delta > 0$, $\exists C = C(n, \lambda, \alpha, \beta, \delta)$
s.t. $$[D^2u]_{\alpha;B^+_{1/2}} \leq \delta[D^2u]_{\alpha;B^+_{1/2}} + C(|u|_{2;B_1^+} + |f|_{0,\alpha;B_1^+}).$$

Proof of claim: Argue by contradiction.
As before, up to a subsequence $\exists x_k, y_k \in B_1^+$, $u_k \in C^{2,\alpha}(B_1^+)$ and solve $L_k u_k = f_k \in C^{0,\alpha}(B_1^+)$ &
$$[D^2u_k]_{\alpha;B^+_{1/2}} > \delta[D^2u]_{\alpha;B_1^+} + k(|u_k|_{2;B_1^+} + |f|_{0,\alpha;B_1^+})$$
& $$\left|\frac{D^2_{ln}u_k(x_k) - D^2_{ln}u_k(y_k)}{|x_k - y_k|^\alpha}\right| > 1/2[D^2u]_{\alpha;B^+_{1/2}}$$
Then, as before $\rho_k := |x_k - y_k| \to 0$ as $k \to \infty$.

We have two cases:

either (I) $\limsup_{k\to\infty} \frac{\operatorname{dist}(x_k, S_1)}{\rho_k} = \infty$

or (II) $\limsup_{k\to\infty} \frac{\operatorname{dist}(x_k, S_1)}{\rho_k} = \mu < \infty.$

LECTURE 11

Proof of Thm 3.4 (continued)

Claim: $\forall \delta>0, \exists C$ s.t.

$$[D^2u]_{\alpha;B^+_{1/2}} \le \delta\cdot[D^2u]_{\alpha;B_1^+} + C\cdot\left(|u|_{0;B_1^+}+|f|_{0,\alpha;B_1^+}\right)$$

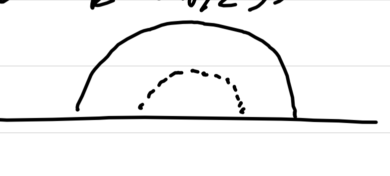
Two cases:

(1) either $\limsup_{k\to\infty} \frac{dist(x_k,S_1)}{\rho_k}=\infty$

(2) or $\limsup_{k\to\infty} \frac{dist(x_k,S_1)}{\rho_k}=\mu<\infty$.

Case 1: Here $\forall R>0$ & k suff. large

$1/2 \ge dist(x_k,S_1) > R\cdot\rho_k$ (as $x_k\in B^+_{1/2}$),



so have $B_{R\rho_k}(x_k)\subset B_1^+$

Set (as before)

$$\tilde u_k(x) = \frac{u_k(x_k+\rho_k x) - q_k(x)}{\rho_k^{2+\alpha}[D^2u_k]_{\alpha;B_1^+}}$$

where $q_k(x) = u_k(x_k) + \rho_k x^i\partial_i u_k(x_k) + \frac{1}{2}\rho_k^2 x^i x^j \partial_i\partial_j u_k(x_k)$.

Then $\tilde u_k$ defined in $B_R(0)$, and $|\tilde u_k|_{2,\alpha;B_R(0)} \le C(R)$ using Arzela-Ascoli proof goes through as in Thm 3.1.

Case 2: Here $\limsup_{k\to\infty} \frac{dist(x_k,S_1)}{\rho_k}=\mu<\infty$

let $z_k = proj_{\{x^n=0\}}(x_k)$, i.e.

$z_k = (x_k^1,\dots,x_k^{n-1},0)$. As before, look at $C^{2,\alpha}$ norm, i.e. define

$$\tilde u_k(x) = \frac{u_k(z_k+\rho_k x) - \tilde q_k(x)}{\rho_k^{2+\alpha}[D^2u_k]_{\alpha;B_1^+}}$$

where $\tilde q_k(x) = u_k(z_k) + \rho_k x^i \partial_i u_k(z_k) + \frac{\rho_k^2}{2} x^i x^j \partial_i\partial_j u_k(z_k)$

$= \rho_k\partial_n u_k(z_k)x^n + \frac{\rho_k^2}{2}\partial_{nj}u_k(z_k)x^n x^j$

because $u|_{S_1}=0$ and $\partial_i u|_{S_1}=0$ $\forall i\neq n$.

In particular, as before, have

$$[D^2\tilde u_k]_{\alpha;B^+_R(0)}\le 1$$

and for any $R>0$,

$|\tilde u_k|_{2;0;B_R^+(0)} \le C(R)$ for k suff. large. Also, $\boxed{\tilde u_k|_{S_R}=0}$ since on $\{x^n=0\}$, $\tilde q_k(x)=0$.

Set $\xi_k = \frac{x_k-z_k}{\rho_k}$, $\eta_k = \frac{y_k-z_k}{\rho_k}$.

Then for k suff. large, $|\xi_k|\le 2\mu$ and

$$|\eta_k| \le \frac{|x_k-y_k|+|x_k-z_k|}{\rho_k} \le 1+2\mu.$$

So both sequences are bounded (and lie in compact subsets of $\overline{\mathbb{R}^n_+}$), so can find convergent subsequences, $\xi_k\to\xi$, $\eta_k\to\eta$.

Then

$$D^2_{lm}\tilde u_k(\xi_k) = \frac{D^2_{lm}u_k(x_k)-D^2_{lm}u_k(z_k)}{\rho_k^\alpha[D^2u_k]_{\alpha;B_1^+}}$$

and

$$D^2_{lm}\tilde u_k(\eta_k) = \frac{D^2_{lm}u_k(y_k)-D^2_{lm}u_k(z_k)}{\rho_k^\alpha[D^2u_k]_{\alpha;B_1^+}}$$

So

$$|D^2_{lm}\tilde u_k(\xi_k) - D^2_{lm}\tilde u_k(\eta_k)| = \frac{|D^2_{lm}u_k(x_k) - D^2_{lm}u_k(y_k)|}{\rho_k^\alpha[D^2u_k]_{\alpha;B_1^+}}$$

$$\ge \frac{1/2\,[D^2u_k]_{\alpha;B^+_{1/2}}}{[D^2u_k]_{\alpha;B_1^+}} \ge \frac{\delta}{2}>0 \quad (*)$$

using contradiction assumption of pf of claim

Then by Arzela-Ascoli (AA) we obtain $v\in C^{2,\alpha}(\mathbb{R}^n_+\cup\{x^n=0\})$ s.t. $\tilde u_k\to v$ in C^2 on compact subsets of $\mathbb{R}^n_+\cup\{x_n=0\}$, as before, v satisfies

$$\tilde a^{ij}D^2_{ij}v=0 \quad \text{— elliptic}$$

and $\tilde a^{ij}$ constant in $\mathbb{R}^n_+$ & also $v|_{\{x^n=0\}}=0$. Then, again as before, by rotation and scaling, we get that $\exists\, w\in C^2(\overline{H})$, ($H=\{x^n>0\}$).

s.t.

$$\begin{cases}\Delta w=0 & \text{on } H\\ w|_{\partial H}=0\end{cases}$$

By making an odd reflection in ∂H (see below), we can extend w to a harmonic fn $\tilde w$ on $\mathbb{R}^n$, with $[D^2\tilde w]_{\alpha;\mathbb{R}^n}<\infty$. But then, this implies that $D^2\tilde u$ is harmonic and grows sublinearly, hence $\tilde w$ is constant (by Liouville). But then this contradicts $(*)$ after taking it to the limit and so we are done with the claim. $\square$

To finish the proof of the theorem, by interpolation and scaling, (just as in Thm 3.1), we have for any $B_\rho(y)\subset B_1$ with $y\in\{x^n=0\}$, we have

$$\rho^{2+\alpha}[D^2u]_{\alpha;B^+_{\rho/2}(y)} \le \delta\cdot\rho^{2+\alpha}[D^2u]_{\alpha;B^+_\rho(y)} + C\cdot\left(|u|_{0;B_1^+}+|f|_{0,\alpha;B_1^+}\right)$$

Also, by the interior estimate, for any $B_\rho(y)$ s.t. $\overline{B_\rho(y)}\subset B_1^+$, have

$$\rho^{2+\alpha}[D^2u]_{\alpha;B_{\rho/2}(y)} \le C\left(|u|_{0;B_1^+}+|f|_{0,\alpha;B_1^+}\right).$$

Then the conclusion follows from boundary absorbing lemma

Proposition 3.5: (Reflection Principle for Harmonic Functions)

Let Ω^+ be an open subset of $\mathbb{R}^n_+$ and let $T=\partial\Omega^+\cap\{x^n=0\}$. Let Ω^- be the reflection of Ω^+ in $\{x^n=0\}$, i.e.

$$\Omega^-=\{(x',-x^n):(x',x^n)\in\Omega_+\}.$$

Let $v\in C^2(\Omega^+)\cap C^0(\Omega^+\cap T)$ & $\bar v$ be the odd reflection of v in T, i.e.

$$\bar v:\Omega^+\cup T\cup\Omega^-\to\mathbb{R}$$

$$\bar v(x',x^n)=\begin{cases}v(x',x^n), & (x',x^n)\in\Omega_+\\ -v(x',-x^n), & (x',-x^n)\in\Omega_-\end{cases}$$

Then if $\Delta v=0$ in Ω_+ and $v|_T=0$, then $\bar v\in C^2((\Omega_+\cup T\cup\Omega_-)^\circ)$ & $\Delta\bar v=0$.

Proof: (use MVP, ES2)

Remark: This is trivial if $T=\emptyset$, as then Ω^+ & Ω^- disjoint. Important part is C^2 across T.

Proposition 3.6: (Absorbing Lemma, Boundary Version).

Given $\theta\in(0,1)$, $\mu\in\mathbb{R}$, then $\exists\,\delta=\delta(n,\theta,\mu)$ and $C=C(n,\theta,\mu)$ s.t.: if $R>0$,

$$\mathcal{B}=\{B_\rho(y)\subset B_R^+(0)\}$$

$$\mathcal{B}^+=\{B^+_\rho(y): y^n=0,\ B^+_\rho(y)\subset B^+_R(0)\}$$

and $S:\mathcal{B}\cup\mathcal{B}^+\to\mathbb{R}_{\ge0}$, sub-additive function satisfying:

$$\rho^\mu S(B^+_{\theta\rho}(y))\le\delta\cdot\rho^\mu S(B^+_\rho(y))+\gamma$$

for all $B^+_\rho(y)\in\mathcal{B}^+$ and

$$\rho^\mu S(B_{\theta\rho}(y))\le\delta\rho^\mu S(B_\rho(y))+\gamma$$

for $B_\rho(y)\in\mathcal{B}$. Then

$$R^\mu S(B^+_{\theta R}(0))\le C\gamma$$

Proof: ES 2 $\square$

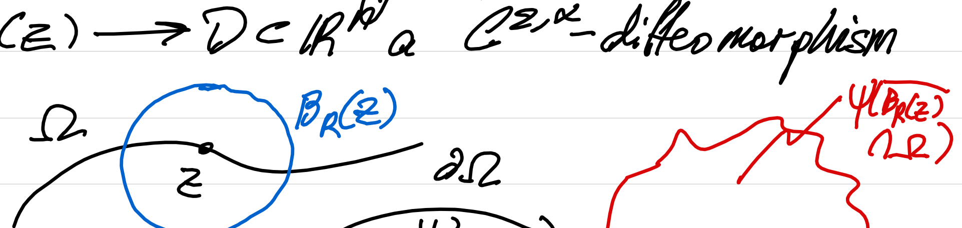
LECTURE 12

Shorthand: write "hypothesis (H)" for: "Suppose $\Omega \subset \mathbb{R}^n$ is a <u>bounded</u> domain & $\alpha \in (0,1)$. Suppose $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$ are s.t.
$$|a^{ij}|_{0,\alpha;\Omega} + |b^i|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \le \beta.$$
and suppose that $\exists\, \lambda > 0$ s.t.
$$a^{ij}(x)\xi^i\xi^j \ge \lambda|\xi|^2 \quad \forall x \in \Omega, \xi \in \mathbb{R}^n.$$
As always, $L = a^{ij}\partial_i\partial_j + b^i\partial_i + c$."

<u>Theorem 3.7</u> (Boundary Schauder Estimates in General Domains)
Suppose (H) holds, Ω is $C^{2,\alpha}$ domain, then $\exists\, \varepsilon = \varepsilon(\Omega) > 0$ s.t. if $u \in C^{2,\alpha}(\bar{\Omega})$, $f \in C^{0,\alpha}(\Omega)$, $\varphi \in C^{2,\alpha}(\bar{\Omega})$ solve
$$\begin{cases} Lu = f, & \text{in } \Omega \\ u = \varphi, & \text{on } \partial\Omega \end{cases}$$
Then $\forall x \in \partial\Omega$.
$$|u|_{2,\alpha;B_\varepsilon(x)\cap\Omega} \le C \cdot \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right).$$

<u>Remark:</u> Need Ω to be $C^{2,\alpha}$ to have any chance of u being $C^{2,\alpha}$ on $\partial\Omega$.

<u>Proof:</u> Pick $z \in \partial\Omega$. By definition, $\exists R > 0$ & $\psi: B_R(z) \to D \subset \mathbb{R}^n$ a $C^{2,\alpha}$-diffeomorphism s.t.



i.e. $\psi(B_R(z) \cap \Omega) \subset \{y^n > 0\}$
& $\psi(B_R(z) \cap \partial\Omega) \subset \{y^n = 0\}$
i.e. ψ "rectifies $\partial\Omega$ near z. Let $x = (x^1, x^2, \dots, x^n)$ be coordinates in Ω & let $y = (y^1, y^2, \dots, y^n)$ be coordinates in D. Let $\tilde{u}(y) = u(\psi^{-1}(y))$ – the pullback of u along ψ^{-1}.
Then
$$\tilde{u}|_{\{y^n=0\}\cap D} = (\varphi \circ \psi^{-1})|_{\{y^n=0\}\cap D} =: \tilde{\varphi}$$

To apply Theorem 3.4 (Unit boundary Schauder) need to find PDE satisfied by $\tilde{u}$ & show it satisfies the hypotheses.
Note $u(x) = \tilde{u}(\psi(x))$, so
$$\partial_{x_i} u = \partial_{y_k}\tilde{u}\,\frac{\partial\psi^k}{\partial x^i}, \text{ so}$$
$$\partial^2_{x_i x_j} u = \partial^2_{y_\ell y_k}\tilde{u}(\psi(x))\frac{\partial\psi^k}{\partial x^i}\frac{\partial\psi^\ell}{\partial x^j} + \partial_{y_k}\tilde{u}(\psi(x))\frac{\partial^2\psi^k}{\partial x^i \partial x^j}$$
Hence can find the coefficients of the new PDE explicitly:
$$A^{\ell k}\partial^2_{y_k}\tilde{u} + B^\ell\partial_{y_\ell}\tilde{u} + C\tilde{u} = \tilde{f} \text{ on } D$$
$\tilde{u} = \tilde{\varphi}$ on $D \cap \{y^n = 0\}$ where
$$A^{\ell k} = a^{ij}\frac{\partial\psi^k}{\partial x^i}\frac{\partial\psi^\ell}{\partial x^j}, \quad B^\ell = \frac{\partial\psi^k}{\partial x^i}b^i + a^{ij}\frac{\partial^2\psi^k}{\partial x^i\partial x^j}$$
$$C = c \circ \psi^{-1}, \qquad \tilde{f} = f \circ \psi^{-1}$$

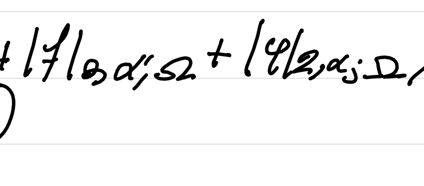
Rescale: choose $\sigma > 0$ s.t. $B_\sigma(\psi(z)) \subset D$ (as D open). Want to apply Thm 3.4 with $\bar{u}(y) = \tilde{u}(\psi(z) + \sigma y)$. We will get
$$(\dagger)\ |\tilde{u}|'_{2,\alpha;B^+_{\sigma/2}(\psi(z))} \le C \cdot \Big(|\tilde{u}|_{0;B^+_\sigma(\psi(z))} + \sigma^2|\tilde{f}|_{0,\alpha;B^+_\sigma(\psi(z))} + |\tilde{\varphi}|'_{2,\alpha;B^+_\sigma(\psi(z))}\Big)$$
for some $C = C(n, \lambda, \alpha, \beta, \psi)$. To apply Thm 3.4, need to check assumptions:
(a) coefficients are bounded, (b) strict ellipticity.
(a) $|A^{\ell k}|'_{0,\alpha;B^+_\sigma(\psi(z))} + |B^\ell|'_{0,\alpha;B^+_\sigma(\psi(z))} + |C|_{0,\alpha;B^+_\sigma(\psi(z))} \le \mu(\psi)\cdot\beta$
&
(b) For this, note
$$A^{\ell k}(y)\xi^\ell\xi^k \overset{\text{check}}{=} a^{ij}\,\partial_i(\xi\cdot\psi)\,\partial_j(\xi\cdot\psi)$$
$$\underset{(a^{ij}\text{ elliptic})}{\ge} \lambda|D(\xi\cdot\psi)|^2\big|_{\psi^{-1}(y)}$$
$$\ge \lambda c(\psi)|\xi|^2$$
this follows from:
$$\xi\cdot y = \xi\cdot\psi(\psi^{-1}(y))$$
$$\Longrightarrow \xi = D(\xi\cdot\psi)\big|_{\psi^{-1}(y)}\cdot D\psi^{-1}\big|_y \quad \text{(chain rule)}$$
$$\Longrightarrow |\xi| \le |D(\xi\cdot\psi)|\big|_{\psi^{-1}(y)}\cdot\underbrace{\|D\psi^{-1}\|_0}_{c(\psi)^{-1/2} \in (0,\infty)}.$$
Can check (a) similarly. So transforming RHS of $(\dagger)$ to $\tilde{u} \to u$, $\tilde{f} \to f$, $\tilde{\varphi} \to \varphi$ have
$$|\tilde{u}|'_{2,\alpha;B^+_{\sigma/2}(\psi(z))} \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right).$$
where $C = C(n, \lambda, \alpha, \beta, \psi, \sigma)$.
Pick $r > 0$, $r = r(z)$, s.t.
$B_r(z) \subset \psi^{-1}(B_{\sigma/2}(\psi(z)))$. Then from above:
$$(\dagger\dagger)\ |u|_{2,\alpha;B_r(z)\cap\Omega} \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)$$

All this was done for a fixed $z \in \partial\Omega$ so have $\psi = \psi_z$, $\sigma = \sigma_z$, $C = C_z$.
We finish with a compactness argument.
Clearly $\partial\Omega \subset \bigcup_{z\in\partial\Omega} B_{r(z)/2}(z)$, by compactness $\exists$ a finite subcover
$$\partial\Omega \subset \bigcup_{j=1}^N B_{r(z_j)/2}(z_j)$$
Then let $\varepsilon = \min_{1\le j\le N}\left\{\frac{r(z_j)}{2}\right\}$ &
$C = \max_{1\le j\le N}\{C_{z_j}\}$. Then for <u>any</u> $x \in \partial\Omega$
$$|u|_{2,\alpha;B_\varepsilon(x)\cap\Omega} \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)$$
□

<u>§3.3: Global Schauder Estimates</u>

Can combine interior & boundary estimates:
<u>Theorem 3.8:</u> (Global Schauder Estimates)
Suppose (H) holds. Suppose Ω is a $C^{2,\alpha}$ domain. Then if $u \in C^{2,\alpha}(\Omega)$, $f \in C^{0,\alpha}(\Omega)$, $\varphi \in C^{2,\alpha}(\Omega)$ satisfy
$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$
then
$$|u|_{2,\alpha;\Omega} \le C \cdot \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)$$
where $C = C(n, \lambda, \alpha, \beta, \Omega)$

<u>Proof:</u> Let $\varepsilon = \varepsilon(\Omega)$ be as in the Boundary Schauder on General Domains.
Then let: $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge \frac{\varepsilon}{4}\}$



Then by interior estimates, have
$|u|_{2,\alpha;\Omega_\varepsilon} \le C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega})$
Then note that $\Omega \setminus \Omega_\varepsilon \subset \bigcup_{x\in\partial\Omega} B_{\varepsilon/4}(x)$

Therefore $\forall x \in \Omega$
- <u>either</u> $x \in \Omega_\varepsilon$, when
  $|u(x)| + |Du(x)| + |D^2u(x)| \le C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega})$
- <u>or</u> $x \in \Omega \setminus \Omega_\varepsilon$, when $B_{\varepsilon/4}(y)$ contains x for some $y \in \partial\Omega$. By <u>Thm 3.7</u>:
$$|u(x)| + |Du(x)| + |D^2u(x)| \le |u|_{2,\alpha;B_{\varepsilon/4}(y)\cap\Omega} \le C\cdot\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right).$$
So in both cases:
$$|u|_{2;\Omega} \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)$$
<u>To be continued...</u>

LECTURE 13

⊛ Global Schauder Estimates

↓

⊛ Solvability of the "Dirichlet problem"

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

⇝ Quasilinear Theory (2nd order)

De Giorgi – Nash – Moser (a priori estimate)
Application to <u>minimal surface equation</u>

(Proof of Global Schauder Estimate continued).

$Lu \equiv a^{ij} D^2_{ij} u + b^i D_i u + cu$

<u>Hyp(H)</u>: $\alpha \in (0,1)$, Ω, $\Omega \subset \mathbb{R}^n$ bdd domain.
$a^{ij}, b^i, c \in C^{0,\alpha}(\bar\Omega)$, with
$$\sum_{i,j} |a^{ij}|_{0,\alpha;\Omega} + \sum_i |b^i|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \le \beta$$

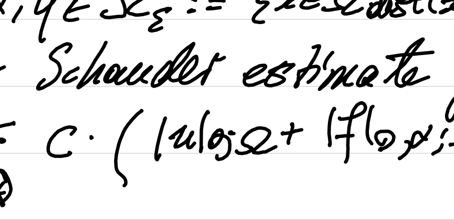
Strict ellipticity: $a^{ij}(x)\xi^i\xi^j \ge \lambda|\xi|^2$ $\forall \xi \in \mathbb{R}^n$, $\forall x \in \Omega$, where $\lambda > 0$ constant.

<u>Thm 3.8</u> (Global <u>Schauder</u> estimate): Suppose that $\Omega \subset \mathbb{R}^n$ is a bdd $C^{2,\alpha}$ domain: if hyp(H) holds, and if $u \in C^{2,\alpha}(\bar\Omega)$, $f \in C^{0,\alpha}(\bar\Omega)$, $\varphi \in C^{2,\alpha}(\bar\Omega)$ satisfy

$$(1)\quad \begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

then $|u|_{2,\alpha;\Omega} \le C\left(|u|_{0,\alpha;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)$
$C = C(n, \lambda, \alpha, \beta, \Omega)$.

<u>Proof</u>: (continued)
<u>Last lecture</u>: $|u|_{2;\Omega} \le C_1\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)$,
$C_1 = C_1(n, \lambda, \alpha, \beta, \Omega)$.
Remains to bound $[D^2 u]_{\alpha;\Omega}$.
Let $x, y \in \Omega$, $x \neq y$,



Let ε be as in Thm 3.7. Suppose $|x-y| < \varepsilon/4$,
⇝ two subcases: $x, y \in \Omega_\varepsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon/4\}$
in this case, interior Schauder estimate gives
$$\frac{|D^2_{ij}u(x) - D^2 u(y)|}{|x-y|^\alpha} \le C \cdot \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right). \quad ⊛$$

If $x \in \Omega\setminus\Omega_\varepsilon$ or $y \in \Omega\setminus\Omega_\varepsilon$
then $x, y \in B_{\varepsilon/2}(z)$, $z \in \partial\Omega$. Then Thm 3.7 gives ⊛

If $|x-y| \ge \varepsilon/4$:
$$\frac{|D^2_{ij}u(x) - D^2_{ij}u(y)|}{|x-y|^\alpha} \le (\varepsilon/4)^{-\alpha}\left(|D_{ij}u(x)| + |D_{ij}u(y)|\right)$$
$$\le 2\cdot(\varepsilon/4)^{-\alpha}\cdot|u|_{2;\Omega}$$
$$\le (\varepsilon/4)^{-\alpha}\cdot c\cdot\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)$$
by (1) □

§4: Solvability of the Dirichlet problem

Given $a^{ij}, b^i, c \in C^{0,\alpha}(\bar\Omega)$, the Dirichlet problem for L is: given $f \in C^{0,\alpha}(\bar\Omega)$, $\varphi \in C^{2,\alpha}(\bar\Omega)$, does there exist a solution $u \in C^2(\bar\Omega)$ to:
$$\begin{cases} Lu = f, & \text{in } \Omega \\ u = \varphi, & \text{on } \partial\Omega \end{cases} \quad (DP)$$
if exists, is it unique?

<u>Thm 4.1</u>: Let $\alpha \in (0,1)$, $\Omega \subseteq \mathbb{R}^n$ a bdd $C^{2,\alpha}$ domain. Suppose $a^{ij}, b^i, c \in C^{0,\alpha}(\bar\Omega)$, <u>$c \le 0$ in Ω</u>, (necessary, see E§2) ($a^{ij}(x)\xi^i\xi^j \ge \lambda|\xi|^2$, $\lambda > 0$ cnst, $\forall x \in \Omega$, $\xi \in \mathbb{R}^n$)

(1) For any given $f \in C^{0,\alpha}(\bar\Omega)$, $\varphi \in C^{2,\alpha}(\bar\Omega)$, the Dirichlet problem $Lu = f$ in Ω, $u = \varphi$ on $\partial\Omega$ has a solution $u \in C^{2,\alpha}(\bar\Omega)$

⟺

(2) For any given $f \in C^{0,\alpha}(\bar\Omega)$, $\varphi \in C^{2,\alpha}(\bar\Omega)$, the (DP) $\Delta u = f$ in Ω, $u = \varphi$ on $\partial\Omega$ has a solution $u \in C^{2,\alpha}(\bar\Omega)$.

<u>Proof</u>: By considering $u - \varphi$ in place of u, it suffices to establish the equivalence for the case $\varphi \equiv 0$
$$\left[\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \Longleftrightarrow \begin{cases} Lv = f - L\varphi \in C^{0,\alpha}(\bar\Omega) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega,\ v = u - \varphi \end{cases}\right]$$

So assume $\varphi = 0$
Note $C^{2,\alpha}_0(\bar\Omega) := \{v \in C^{2,\alpha}(\bar\Omega) : v = 0 \text{ on } \partial\Omega\}$ is a closed subspace of $C^{2,\alpha}(\bar\Omega)$ wrt usual norm, hence Banach.
Consider 1-parameter family of operators:
$$L_t : C^{2,\alpha}_0(\bar\Omega) \to C^{0,\alpha}(\bar\Omega).$$
$$L_t = tLu + (1-t)\Delta u$$
$$\text{so } L_0 = \Delta u,\ L_1 = L.$$

$L_t u \equiv a^{ij}_t D_{ij} u + b^i_t D_i u + c_t u$,
$a^{ij}_t = t a^{ij} + (1-t)\delta_{ij}$, $b^i_t = t b^i$, $c_t = tc$

Let $\beta = \sum |a^{ij}|_{0,\alpha;\Omega} + \sum |b^i|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega}$
$\Rightarrow \sum |a^{ij}_t|_{0,\alpha;\Omega} + \sum |b^i_t|_{0,\alpha;\Omega} + |c_t|_{0,\alpha;\Omega} \le \max\{1, \beta\}$ $\forall t \in [0,1]$.
and similarly $a^{ij}_t \xi^i \xi^j \ge \min\{1, \lambda\}|\xi|^2$ $\forall t \in [0,1]$.

Global Schauder Estimate (Thm 3.8) ⇒
$$|u|_{2,\alpha;\Omega} \le C\cdot\left(|u|_{0,\alpha;\Omega} + |L_t u|_{0,\alpha;\Omega}\right)$$
$\forall u \in C^{2,\alpha}_0(\bar\Omega)$, $C = C(n, \lambda, \alpha, \beta, \Omega)$.

LECTURE 14

<u>Proof</u> (Thm 4.1):

$$L_t : C^{2,\alpha}_0(\overline{\Omega}) \longrightarrow C^{0,\alpha}(\overline{\Omega}),$$
$$L_t = tL + (1-t)\Delta.$$

Global Schauder $\Longrightarrow |u|_{2,\alpha;\Omega} \le C_1 \cdot (|u|_{0;\Omega} + |L_t u|_{0,\alpha;\overline{\Omega}})$
$\forall u \in C^{2,\alpha}_0(\overline{\Omega})$. $C_1 = C_1(n,\lambda,\alpha,\beta,\Omega)$,
(indep. of t and u).

Since $c \le 0$, by the max. principle a priori estimate (Thm 2.9?):

$$|u|_{0;\Omega} \le C_2 \, |L_t u|_{0,\alpha;\overline{\Omega}}, \quad C_2 = C_2(n,\lambda,\alpha,\beta,\Omega).$$

So $\boxed{|u|_{2,\alpha;\overline{\Omega}} \le C \, |L_t u|_{0,\alpha;\overline{\Omega}},}$ $\forall u \in C^{2,\alpha}_0(\overline{\Omega})$
$\forall t \in [0,1]$, $C = C(n,\lambda,\alpha,\beta,\Omega)$.

This says L_t is injective. Solvability of $L_t u = f$ in $C^{2,\alpha}_0(\overline{\Omega})$ is equivalent to surjectivity of L_t ($\Leftrightarrow$ bijectivity of L_t).
We will show if L_t is surjective for some $t \in [0,1]$, then it is surjective for all $t \in [0,1]$.

Let $s \in [0,1]$ and suppose L_s is bijective:
The estimate above can be written as

$$|L_s^{-1}(g)|_{2,\alpha;\overline{\Omega}} \le C \cdot |g|_{0,\alpha;\overline{\Omega}} \quad \forall g \in C^{0,\alpha}(\overline{\Omega})$$

Fix $f \in C^{0,\alpha}(\overline{\Omega})$

$$\begin{aligned} L_t u = f &\Longleftrightarrow L_s u + (L_t - L_s)u = f \\ &\Longleftrightarrow u + L_s^{-1}((L_t - L_s)u) = L_s^{-1} f \\ &\Longleftrightarrow u = \underbrace{L_s^{-1} f + L_s^{-1}((L_s - L_t)u)}_{T_t u} \end{aligned}$$

<u>Claim:</u> $T_t : C^{2,\alpha}_0(\overline{\Omega}) \to C^{2,\alpha}_0(\overline{\Omega})$ is a contraction mapping provided $|t-s| \le \gamma$, where $\gamma = \gamma(n,\alpha,\beta,\lambda,\Omega)$.

<u>Proof of the claim:</u> For $u, v \in C^{2,\alpha}_0(\overline{\Omega})$,

$$\begin{aligned} |T_t u - T_t v|_{2,\alpha;\overline{\Omega}} &= |L_s^{-1}((L_s - L_t)(u-v))|_{2,\alpha;\overline{\Omega}} \\ &= |s-t| \cdot |L_s^{-1}(L-\Delta)(u-v)|_{2,\alpha;\overline{\Omega}} \\ &\le C \cdot |s-t| \cdot |(L-\Delta)(u-v)|_{0,\alpha;\overline{\Omega}}, \quad \text{direct computation.} \\ &\le \tilde{C} \cdot |s-t| \cdot |(u-v)|_{2,\alpha;\overline{\Omega}} \end{aligned}$$

So if $|s-t| \le \underbrace{\frac{1}{2\tilde{C}}}_{\gamma}$, then T_t is a contraction. So by the contraction mapping principle, T_t has a (unique) fixed point $u \in C^{2,\alpha}_0(\overline{\Omega})$.

If solvability of $L_s u = f$ for $u \in C^{2,\alpha}_0(\overline{\Omega})$ holds for some $s \in [0,1]$, then solvability of $L_t u = f$ for $u \in C^{2,\alpha}_0(\overline{\Omega})$ holds for all $t \in [s-\gamma, s+\gamma]$.
By breaking $[0,1]$ into intervals of length 2γ, (and applying this conclusion in each subinterval, we arrive at the conclusion of the thm $\square$

<u>Rmk:</u> The method of proof is called the <u>continuity method</u>. The next main steps of solvability of L:
(i) use Thm 4.1 to prove solvability when $\Omega = B$ a ball.
(ii) Perron's method: "solvability in balls $\Rightarrow$ solvability in general domains".

<u>Prop 4.2:</u> Let $B = B_R(y) \subseteq \mathbb{R}^n$ be any (open) ball. If $f \in C^\infty(\overline{B})$, then $\varphi \in C^\infty(\overline{B})$, then there is a unique function $u \in C^\infty(\overline{B})$ s.t.
$\Delta u = f$ in B, $u = \varphi$ in ∂B.
<u>Proof (sketch):</u> After reducing to $\Delta v = f - \Delta\varphi$, $v = 0$ on ∂B, ($v = u - \varphi$). By Riesz rep. theorem, $\exists$ weak solution $v \in W^{1,2}_0(B)$. Regularity theory (difference quotient argument) $\Rightarrow v \in C^\infty(\overline{B})$.
(See "Analysis of PDE" last term).

Generalise this to the case $f \in C^{0,\alpha}(\overline{B})$, $\varphi \in C^0(\overline{B})$ or ($\varphi \in C^{2,\alpha}(\overline{B})$).

<u>Prop 4.3:</u> Let $B = B_R(y) \subseteq \mathbb{R}^n$. If $f \in C^{0,\alpha}(\overline{B})$ and $\varphi \in C^0(\overline{B})$, then $\exists!\, u \in C^{2,\alpha}(B) \cap C^0(\overline{B})$ s.t. $\Delta u = f$ in B, $u = \varphi$ on ∂B. If $\varphi \in C^{2,\alpha}(\overline{B})$, then $u \in C^{2,\alpha}(\overline{B})$.

<u>Proof:</u> Idea is to mollify f, φ to get smooth data, use the Prop 4.2 to solve for these smooth approximations and then find a limit.

$$\eta(x) = \begin{cases} c \cdot e^{\frac{1}{|x|^2-1}}, & |x| < 1 \\ 0, & |x| \ge 1 \end{cases}, \quad c \text{ chosen s.t. } \int_{\mathbb{R}^n} \eta = 1$$

Define for $\sigma > 0$, $\eta_\sigma(x) = \sigma^{-n}\eta\left(\frac{x}{\sigma}\right)$
choose $\sigma_k \to 0^+$. Extend f to $\tilde{f} \in C^{0,\alpha}_c(\mathbb{R}^n)$ and φ to $\tilde{\varphi} \in C^0_c(\mathbb{R}^n)$.
Mollify $\tilde{f}, \tilde{\varphi}$:

$$f_k(x) = \int_{\mathbb{R}^n} \tilde{f}(y)\,\eta_{\sigma_k}(x-y)\,dy = \int_{\mathbb{R}^n} \tilde{f}(x-y)\,\eta_{\sigma_k}(y)\,dy$$

$$\varphi_k(x) = \int_{\mathbb{R}^n} \tilde{\varphi}(y)\,\eta_{\sigma_k}(x-y)\,dy = \int_{\mathbb{R}^n} \tilde{\varphi}(x-y)\,\eta_{\sigma_k}(y)\,dy$$

Note that $f_k \to f$, $\varphi_k \to \varphi$ uniformly in $\overline{B}$.
We in fact have: $|f_k|_{0,\alpha;\mathbb{R}^n} \le |\tilde{f}|_{0,\alpha;\mathbb{R}^n}$, $|\varphi_k|_{0;\mathbb{R}^n} \le |\tilde{\varphi}|_{0;\mathbb{R}^n}$ (direct computation).
By <u>Prop 4.2</u>, get $u_k \in C^\infty(\overline{B})$ s.t.
$\Delta u_k = f_k$ in B, $u_k = \varphi_k$ on ∂B.

LECTURE 15

<u>Proof of Prop 4.3 (cont'd)</u>: (to solve $u \in C^{2,\alpha}(B) \cap C^0(\overline{B})$ $\Delta u = f$ in B, $u = \varphi$ on ∂B, for $f \in C^{0,\alpha}(\overline{B})$, $\varphi \in C^0(\overline{B})$.

$$\Delta u_k = f_k \text{ in } B.$$
$$u_k = \varphi_k \text{ on } \partial B$$

$$\Delta(u_k - u_l) = f_k - f_l \text{ in } B,$$
$$u_k - u_l = \varphi_k - \varphi_l \text{ on } \partial B.$$

By the max. principle a priori estimate:
$$|u_k - u_l|_{0;\overline{B}} \le |\varphi_k - \varphi_l|_{0;\partial B} + C \cdot |f_k - f_l|_{0;\overline{B}}$$
$$\to 0 \text{ as } k, l \to \infty$$

So u_k is Cauchy and hence converges uniformly to some $u \in C^0(\overline{B})$. In particular, $u = \varphi$ on ∂B.

Now apply interior Schauder estimate: $\to$ C indep. of k

$\forall \tilde{B} \subset\subset B$, $|u_k|_{2,\alpha;\tilde{B}} \le C \cdot (|u_k|_{0;\overline{B}} + |f_k|_{0,\alpha;\overline{B}})$
$$\le C \cdot (|u|_{0;\overline{B}} + |\tilde{f}|_{0,\alpha;\mathbb{R}^n} + 1)$$
mollification bound.

Passing to a subsequence (without relabelling); $\exists v \in C^{2,\alpha}(\tilde{B})$ s.t. $u_k \to v$ in $C^2(\tilde{B})$
[Arzelà-Ascoli].

Since $u_k \to u$ pointwise $\Rightarrow v = u$ in $\tilde{B}$ and so in particular, $u \in C^{2,\alpha}(\tilde{B})$, by passing to limit in $\Delta u_k = f_k$ in $\tilde{B}$, get $\Delta u = f$ in $\tilde{B}$, $\tilde{B} \subset\subset B$ is arbitrary, so $u \in C^{2,\alpha}(B)$ and $\Delta u = f$ in B.

For the 2^{nd} part when $\varphi \in C^{2,\alpha}(\overline{B})$, repeat the argument [after extending φ to $\tilde{\varphi} \in C_c^{2,\alpha}(\mathbb{R}^n)$, see general extension theorem, Gilbarg-Trudinger, Lemma 6.37], (2nd edition) but use global Schauder estimates in place of interior estimate. □

<u>Prop 4.4</u>: $B \subset \mathbb{R}^n$ a ball, $\alpha \in (0,1)$, $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{B})$, $\underline{c \le 0}$, $Lu \equiv a^{ij} D_{ij} u + b^i D_i u + c$ is strictly elliptic.
Then for any $f \in C^{0,\alpha}(\overline{B})$ and $\varphi \in C^0(\overline{B})$, there exist unique $u \in C^{2,\alpha}(B) \cap C^0(\overline{B})$ s.t. $Lu = f$ in B, $u = \varphi$ on ∂B.

<u>Proof</u>: combine thm 4.1 with the <u>proof</u> of prop. 4.3 □

<u>Perron's method</u>: We'll assume the following for the rest of the section. <u>Hypothesis (H)</u>: $\alpha \in (0,1)$
$\Omega \subset \mathbb{R}^n$ bdded, $\sum |a^{ij}|_{0,\alpha;\Omega} + \sum |b^i|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \le \beta$
$a^{ij}(x)\xi^i\xi^j \ge \lambda|\xi|^2$, $\lambda > 0$ constant, $\underline{c \le 0}$

<u>Observation 1</u>: Fix $f \in C^{0,\alpha}(\Omega)$, and suppose that $u \in C^2(\Omega)$. Then u is a subsolution to $Lu = f$ in Ω (i.e. $Lu \ge f$ in Ω) iff for every ball $B \subset\subset \Omega$ we have that $u \le u_B$, where $u_B \in C^{2,\alpha}(B) \cap C^0(\overline{B})$ is the unique f^e satisfying $Lu_B = f$ in B, $u_B = u$ on ∂B.
(such u_B exists by prop. 4.4). This follows from the weak maximum principle (Existence).

<u>Observation 2</u>: $f \in C^{0,\alpha}(\Omega)$, $\varphi \in C^0(\overline{\Omega})$.
$$S_\varphi \equiv \{v \in C^2(\Omega) \cap C^0(\overline{\Omega}) : Lv \ge f \text{ in } \Omega, \; v \le \varphi \text{ on } \partial\Omega\}$$
Then if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solves $Lu = f$ in Ω, $u = \varphi$ on $\partial\Omega$, then
$$u(x) = \sup_{v \in S_\varphi} v(x)$$
Check both obs. 1 & 2 (Ex. Sheet).

<u>Def^n 1</u>: Let $f \in C^{0,\alpha}(\Omega)$. <u>A function $u \in C^0(\Omega)$ is a subsolution (super-) to $Lu = f$ in Ω</u> if, for every ball with $\overline{B} \subset \Omega$, we have $u \le u_B$ ($u \ge u_B$) in $\overline{B}$.
(where u_B is the unique function in $C^2(B) \cap C^0(\overline{B})$ s.t. $Lu_B = f$ in B, $u_B = u$ on ∂B).

<u>Def^n 2</u>: Let $u \in C^0(\Omega)$ be a sub-solution to $Lu = f$ in Ω. Let $B \subset\subset \Omega$ be a ball, then the <u>L-lift of u wrt B</u> is the function U_B defined by
$$U_B(x) = \begin{cases} u_B(x), & x \in B \\ u(x), & x \in \Omega \setminus B. \end{cases}$$

<u>Lemma 4.5</u>: We have the following:
(i) Let $u, v \in C^0(\overline{\Omega})$, if u is a sub-solution and v is a super-solution to to $Lu = f$ in Ω, and if $u \le v$ on $\partial\Omega$, then $u \le v$ in $\overline{\Omega}$.
(ii) If $u_1, u_2 \in C^0(\overline{\Omega})$ are sub solutions to $Lu = f$ in Ω, then $v(x) = \max\{u_1(x), u_2(x)\}$ is again (continuous and) a subsolution to $Lu = f$.
(iii) If $u \in C^0(\Omega)$ is a subsolution, and $B \subset\subset \Omega$, then the L-lift of u is again a (cts) subsolution. [Exercise in ex. Sheet 3, applications of max. principles].

<u>Define</u> For $\varphi \in C^0(\overline{\Omega})$, $f \in C^{0,\alpha}(\Omega)$ fixed.
$$S_\varphi \equiv \{v \in C^0(\overline{\Omega}) : v \text{ is a subsolution in } \Omega, \; v \le \varphi \text{ on } \partial\Omega\}.$$
and set $u(x) = \sup_{v \in S_\varphi} v(x)$

<u>Thm 4.6</u>: The function u defined as above is well-defined (i.e. $S_\varphi \ne \emptyset$ and $u(x) \in \mathbb{R}$) and we have $u \in C^{2,\alpha}(\Omega)$ and solves $Lu = f$ in Ω.

LECTURE 16

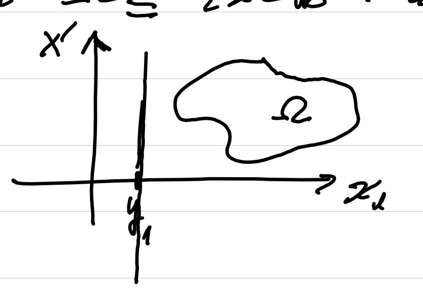
Theorem 4.6: Let hyp(H) hold. Let $f \in C^{0,\alpha}(\bar\Omega)$ and $\varphi \in C^0(\bar\Omega)$. Define $S_\varphi = \{ v \in C^0(\bar\Omega) : v \text{ is a subsolution to } Lu = f \text{ in } \Omega, v \le \varphi \text{ on } \partial\Omega \}$.

$u(x) = \sup_{v \in S_\varphi} v(x) \quad \forall x \in \Omega$. Then $u \in C^{2,\alpha}(\Omega)$ and satisfies $Lu = f$ in Ω.

Remarks: (1) Even though we use the function φ to get the solution u as above, in this theorem, there is no claim about the behaviour of u on approach to $\partial\Omega$.

(2) Once we know $u \in C^2(\Omega)$, we of course have that
$$u(x) = \sup_{\left\{\substack{v \in C^2(\Omega) \\ \text{sub-solution to} \\ Lu = f,\ v \le \varphi}\right\}} v(x).$$
However, the proof of the Theorem (including $u \in C^2(\Omega)$) will crucially depend on Lemma 4.5 (ii), (iii)
[u_1, u_2 subsolutions $\Rightarrow \max\{u_1, u_2\}$ is a subsolution
L-lift of a subsolution is a sub-sol^n].
and they are not valid for the smaller class of C^2 subsolutions. In this sense, the philosophy of the proof is similar to the Hilbert space (variational) approaches to solving PDEs.

Proof: check that $S_\varphi \neq \emptyset$. Pick $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ s.t. $\Omega \subseteq \{ x \in \mathbb{R}^n : x_1 \ge y_1 \}$.



Let $d = \sup_{x \in \Omega} |x - y| < \infty$ (Ω is bounded).

replace with any bound on x_1 ($y_1 \le x_1 \le y_1 + d$).

$$s(x) = -\sup_{\partial\Omega} |\varphi| - (e^{\gamma d} - e^{\gamma(x_1 - y_1)}) \cdot \sup_\Omega |f|,$$
$\leftrightarrow$ (MAYBE WRONG SIGN?)

for γ suff. large constant. By direct calculation
$$Ls = e^{\gamma(x_1 - y_1)} \sup_\Omega |f| (a^{11}\gamma^2 + b^1\gamma + c)$$
$$- c\left(\sup_{\partial\Omega} |\varphi| + e^{\gamma d} \sup_\Omega |f|\right) \quad \text{(check)}.$$
$$(c \le 0) \quad \ge e^{\gamma(x_1 - y_1)} \sup_\Omega |f| (\lambda\gamma^2 + b^1\gamma + c)$$
$$\ge \sup_\Omega |f| \ge f \text{ if } \gamma \text{ is suff. large.}$$

Also, $s \le -\sup_{\partial\Omega} |\varphi| \le \varphi$ on $\partial\Omega$.

Thus, $s \in S_\varphi$, so $S_\varphi \neq \emptyset$. Moreover, if $s_1 = -s$, then $Ls_1 = -Ls \le -\sup_\Omega |f| \le f$ in Ω, $s_1 \ge \varphi$ on $\partial\Omega$ by (**).

So by Lemma 4.5 (i), $v \le s_1 \ \forall v \in S_\varphi$.
In particular, $u(x) \le s_1 \ \forall x \in S_\varphi$, hence u is well-defined.

Fix $z \in \Omega$, and choose $R > 0$ s.t. $\overline{B_R(z)} \subset \Omega$.
By defn of $u(z)$, $\exists v_j \in S_\varphi$ s.t. $v_j(z) \to u(z)$.
Let $\tilde v_j = \max\{v_j, s\} \in S_\varphi$ (Lemma 4.5)

So, $u(z) \ge \tilde v_j(z) \ge v_j(z) \Rightarrow \tilde v_j(z) \to u(z)$ — (1)
and $s \le \tilde v_j \le s_1 (= -s) \Rightarrow \sup_\Omega |\tilde v_j| \le \sup_\Omega |s|$.

Let V_j be the L-lift of $\tilde v_j$ wrt to the ball $B_R(z)$. So we have $LV_j = f$ in $B_R(z)$, $V_j = \tilde v_j$ on $\partial B_R(z)$.
$V_j \in S_\varphi$ (Lemma 4.5 (iii)), and $V_j \ge \tilde v_j$.
$\Rightarrow u(z) \ge V_j(z) \ge \tilde v_j(z) \ (\to u(z))$.
By interior Schauder estimates $\Rightarrow$
$$|V_j|_{2,\alpha; B_{R/2}(z)} \le C \cdot (|V_j|_{0; B_R(z)} + |f|_{0,\alpha; B_R(z)})$$
max. principle estimate $\longrightarrow$
$$\le C (|\tilde v_j|_{0; \partial B_R(z)} + |f|_{0,\alpha; B_R(z)})$$
$$\le C \cdot (\sup_\Omega |s| + |f|_{0,\alpha; B_R(z)}).$$

Arzela-Ascoli $\Rightarrow \exists V \in C^{2,\alpha}(B_{R/2}(z))$ s.t. passing to a subsequence, $V_j \to V$ in $C^2(B_{R/2}(z))$.
In particular $LV = f$ (passing to limit in $LV_j = f$)
By (1), $V(z) = u(z)$.
Claim: $u \equiv V$ in $B_{R/16}(z)$. This will complete the proof, since $V \in C^{2,\alpha}(B_{R/16}(z))$ and solves $LV = f$, and $z \in \Omega$ is arbitrary.

Proof of claim:
Since $u \ge V_j$ (since $V_j \in S_\varphi$), we also have $u \ge V$ in $B_{R/2}(z)$. If claim false, then $\exists z_1 \in B_{R/16}(z)$, s.t. $V(z_1) < u(z_1)$. So $\exists w \in S_\varphi$ s.t. $V(z_1) < w(z_1) \le u(z_1)$. — (2) Let $w_j = \max\{w, V_j\} \in S_\varphi$.
$u \ge w_j \ge V_j$
Let W_j be the L-lift of w_j wrt $B_{R/4}(z_1)$.
By interior Schauder estimate as before,
$\exists W \in C^{2,\alpha}(B_{R/8}(z_1))$ s.t. passing to a subseq.
$W_j \to W$ in $C^2(B_{R/8}(z_1))$.
Have $LW = f$ in $B_{R/8}(z_1)$.
Now, $W_j \ge w_j \ge V_j$ in $B_{R/4}(z_1)$. — (3)
(3) $\Rightarrow W \ge V$ in $B_{R/8}(z_1)$.
($c \le 0$ & strong max. principle) $\Rightarrow L(W - V) = 0$.
By (3) and the fact that $V_j(z) \to u(z)$, we have that $W(z) = V(z)$, but since $z \in B_{R/8}(z_1)$, we have by the SMP that $W \equiv V$ in $B_{R/8}(z_1)$.
By (2), $V(z_1) < w(z_1) \le w_j(z_1) \le W_j(z_1)$
$\qquad$ (defn of w_j) $\qquad$ (L-lift of w_j)
$\Rightarrow (j \to \infty) \quad V(z_1) < W(z_1)$, contradiction.

$\square$

LECTURE 17

Theorem 4.6: Existence of Perron solution, i.e. $u \in C^{2,\alpha}(\Omega)$ solves $Lu = f$, $u(x) = \sup_{v \in S_\varphi} v(x)$, $S_\varphi = \{ v \in C^0(\overline{\Omega}) : v$ is a subsolution to $Lu = f$, $v \leq \varphi$ on $\partial\Omega \}$.

Next goal: Discuss the behaviour of u on approach to $\partial\Omega$. We'll show that under a mild regularity condition on Ω (i.e. if Ω satisfies the exterior sphere condition at every point on $\partial\Omega$), the Perron solution extends to a C^0 function on $\overline{\Omega}$ and satisfies $u(x) = \varphi(x)$ on $\partial\Omega$.

To do this, we need the notion of barriers.

Def^n: Let (H) hold, and let $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^{0,0}(\overline{\Omega})$. Let $x_0 \in \partial\Omega$.

(i) A sequence of functions $w_i^+ \in C^0(\overline{\Omega})$ is an upper barrier at x_0 wrt L, f, φ if

- w_i^+ is a super-solution to $Lu = f$ in Ω with $w_i^+ \geq \varphi$ on $\partial\Omega$, for each i;
- $w_i^+(x_0) \to \varphi(x_0)$ as $i \to \infty$.

(ii) A sequence $w_i^- \in C^0(\overline{\Omega})$ is a lower barrier at x_0 wrt L, f, φ if

- w_i^- is a subsolution to $Lu = f$ in Ω with $w_i^- \leq \varphi$ on $\partial\Omega$, $\forall i$.
- $w_i^-(x_0) \to \varphi(x_0)$ as $i \to \infty$

Prop 4.7: Suppose hyp (H) holds, $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^0(\overline{\Omega})$. Let $x_0 \in \partial\Omega$. Suppose upper and lower barriers at x_0 wrt L, f, φ exist. Then the Perron solution u given by Thm 4.6 has the property that $u(x) \to \varphi(x_0)$ as $x \to x_0$, $x \in \Omega$.

Proof: Let $(w_i^\pm)$ be upper and lower barriers at x_0. Since w_i^+ is a supersolution with $w_i^+ \geq \varphi$ on $\partial\Omega$, we have by lemma 4.5 (i), that $v \leq w_i^+$ in $\overline{\Omega}$ $\forall i$ $\forall v \in S_\varphi \Rightarrow u \leq w_i^+$ in Ω $\forall i$. Also, $w_i^- \leq u$ $\forall i$, since $w_i^- \in S_\varphi$. Since $w_i^\pm(x_0) \to \varphi(x_0)$, and $w_i^\pm \in C^0(\overline{\Omega})$, we get the conclusion. $\square$

Prop 4.8: Suppose hyp (H) holds, $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^0(\overline{\Omega})$. Let $x_0 \in \partial\Omega$. If there exists $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ s.t.:

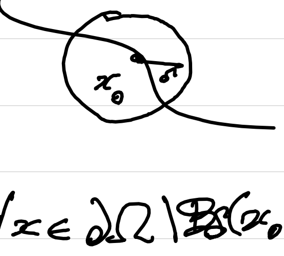
(i) $Lw \leq -1$ in Ω.

(ii) $w(x_0) = 0$

(iii) $w(x) > 0$ $\forall x \in \partial\Omega \setminus \{x_0\}$.

Then, upper and lower barriers exist at x_0 wrt L, f, φ. In fact, for any sequence $\varepsilon_i \to 0^+$, $\exists$ constants k_i s.t. $w_i^\pm(x) = \varphi(x_0) \pm \varepsilon_i \pm k_i w(x)$ define upper and lower barriers.

Proof: Let $\varepsilon > 0$ and choose $r > 0$ s.t. $|\varphi(x) - \varphi(x_0)| < \varepsilon$ $\forall x \in B_r(x_0) \cap \partial\Omega$. Since $\partial\Omega \setminus B_r(x_0)$ is compact, we can find constant l_ε large enough s.t.



$$\begin{cases} l_\varepsilon w(x) \geq \varphi(x) - \varphi(x_0) - \varepsilon \\ l_\varepsilon w(x) \geq -\varphi(x) + \varphi(x_0) - \varepsilon \quad \forall x \in \partial\Omega \setminus B_r(x_0). \end{cases}$$

Set $k_\varepsilon = \max \{ l_\varepsilon, \sup_{x \in \overline{\Omega}} |f(x) - c(x)\varphi(x_0)| \}$

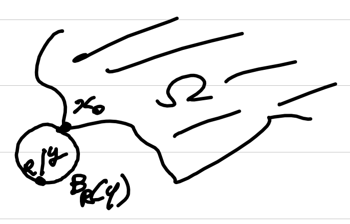
Then we compute $Lw_\varepsilon \leq f$ in Ω where $w_\varepsilon(x) = \varphi(x_0) + \varepsilon + k_\varepsilon w(x)$. So take $\varepsilon_n \downarrow 0$ we get $w_{\varepsilon_n}^+ := w_{\varepsilon_n}$ is an upper barrier.

Similarly, $w_n^- = \varphi(x_0) - \varepsilon_n - k_{\varepsilon_n} \cdot w(x)$, lower barrier. $\square$

LECTURE 18

Prop 4.9: Suppose hyp(H) holds, $f \in C^{0,\alpha}(\bar\Omega)$, $\varphi \in C^0(\partial\Omega)$. Then, if Ω satisfies the exterior sphere condition at $x_0 \in \partial\Omega$, then upper and lower barriers exist at x_0 wrt L, f, φ.

Proof:



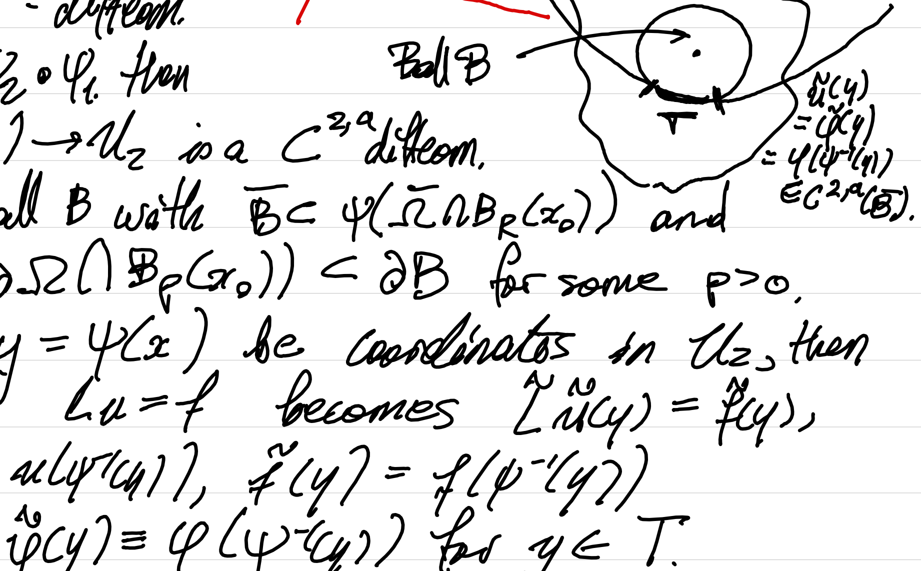
By assumption $\exists B_R(y) \subseteq \mathbb{R}^n$ s.t. $\overline{B_R(y)} \cap \bar\Omega = \{x_0\}$. Let $w(x) = \mu(R^{-\sigma} - |x-y|^{-\sigma})$ for $x \in \bar\Omega$, where $\mu, \sigma > 0$. Then $w \in C^0(\bar\Omega)$, $w(x_0) = 0$. By direct calculation, (check!) that $Lw(x) \le -1$ $\forall x \in \Omega$, provided $\mu, \sigma > 0$ are chosen appropriately. □

Theorem 4.10: Let hyp(H) hold, $f \in C^{0,\alpha}(\bar\Omega)$, and $\varphi \in C^0(\partial\Omega)$. Then there is a unique function $u \in C^{2,\alpha}(\Omega) \cap C^0(\bar\Omega)$ s.t. $Lu = f$ in Ω and $u = \varphi$ on $\partial\Omega$, provided Ω satisfies the exterior sphere condition. (e.g. if Ω is a C^2 domain).

Proof: Let u be given by Thm. 4.6. Then $u \in C^{2,\alpha}(\Omega)$ and satisfies $Lu = f$ in Ω. Then extend u to $\bar\Omega$ by setting $u(x) = \varphi(x)$ $\forall x \in \partial\Omega$. Prop. 4.7–4.9 $\Rightarrow u \in C^0(\bar\Omega)$ □

Thm 4.11: Suppose that hyp. (H) holds, and Ω is a bounded $C^{2,\alpha}$ domain. Then for any $f \in C^{0,\alpha}(\bar\Omega)$, and $\varphi \in C^{2,\alpha}(\bar\Omega)$, there is a unique function $u \in C^{2,\alpha}(\bar\Omega)$ s.t. $Lu = f$ in Ω, $u = \varphi$ on $\partial\Omega$.

Proof:



$\psi_1 : B_R(x_0) \to \mathcal{U}_1$, $C^{2,\alpha}$ diffeomorphism

$\psi_2 : \mathcal{U}_1 \to \mathcal{U}_2$.

both $C^{2,\alpha}$-diffeom.

Let $\psi = \psi_2 \circ \psi_1$ then $\psi : B_R(y) \to \mathcal{U}_2$ is a $C^{2,\alpha}$ diffeom. s.t. $\exists$ ball B with $\bar B \subset \psi(\bar\Omega \cap B_R(x_0))$ and $T \equiv \psi(\partial\Omega \cap B_\rho(x_0)) \subset \partial B$ for some $\rho > 0$. Letting $y = \psi(x)$ be coordinates in $\mathcal{U}_2$, then equation $Lu = f$ becomes $\tilde L \tilde u(y) = \tilde f(y)$, $\tilde u(y) = u(\psi^{-1}(y))$, $\tilde f(y) = f(\psi^{-1}(y))$, $\tilde u(y) = \tilde\varphi(y) \equiv \varphi(\psi^{-1}(y))$ for $y \in T$. Now solve the problem

$$\begin{cases} \tilde L \tilde v = \tilde f & \text{in } B \\ \tilde v = \tilde u & \text{on } \partial B. \end{cases} \quad (*)$$

Get $\tilde v \in C^{2,\alpha}(B) \cap C^0(\bar B)$.

By adapting the same mollification + compactness argument used to prove Theorem 4.3, we also get that $v \in C^{2,\alpha}(B \cup T)$. On the other hand, $\tilde L(\tilde v - \tilde u) = 0$ in B, $\tilde v - \tilde u = 0$ on ∂B, $v - \tilde u \in C^0(\bar B)$. By the weak max. principle, $\tilde v \equiv \tilde u$ on $\bar B$. So in particular $\tilde u \in C^{2,\alpha}(B \cup T)$ so $u \in C^{2,\alpha}(\bar\Omega)$. □

Fredholm Alternative:

V a normed space, and $T : V \to V$ a compact linear map. Then either (i) the equation $x + Tx = 0$ has a non-zero solution $x \in V$ or

(ii) for any given $y \in V$, there is a unique $x \in V$ s.t. $x + Tx = y$.

Proof: Omitted, see Gilbarg & Trudinger, chapter 5 □

(*) Can extend $\tilde u \in C^0(\partial B) \cap C^{2,\alpha}(T')$, $T' \subset\subset T$ to $u^+ \in C^0(\bar B) \cap C^{2,\alpha}(G)$, G is some open nbhd of T'. (See Gilbarg and Trudinger page 137). Then, proceed by mollifying extensions of $\tilde f, u^+$, namely $f_n, u_n \in C^\infty(\bar B)$. Have $f_n \to \tilde f$ and $u_n \to \tilde u$ uniformly in $\bar B$, and

(shrinking G if necessary)
$$\begin{cases} |f_n|_{0,\alpha;\bar B} \le |\tilde f|_{0,\alpha;B} \\ |u_n|_{0;\bar B} \le |u^+|_{0;\bar B} \\ |u_n|_{2,\alpha;G\cap B} \le |u^+|_{2,\alpha;G\cap B} \ \forall n \gg 1 \end{cases}$$

Consider now $v_n \in C^\infty(\bar B)$:

$$\begin{cases} \tilde L v_n = f_n & \text{in } \bar B \\ v_n = u_n & \text{on } \partial B. \end{cases}$$

The usual hyp. (H) holds for $\tilde L \Rightarrow$ apply boundary Schauder estimates near the boundary to get: $\exists \varepsilon > 0$ s.t.

$$|v_n|_{2,\alpha;\underbrace{B_\varepsilon(\psi(x_0)) \cap B}_{\subseteq \tilde G := G \cap B}} \le C \cdot (|v_n|_{0;B} + |f_n|_{0,\alpha;B} + |u_n|_{2,\alpha;G})$$

(WMP a priori bound) $\le C(|u_n|_{0;\partial B} + |f_n|_{0,\alpha;B} + |u_n|_{2,\alpha;G})$

(mollification bound) $\le C(|\tilde u|_{0;\partial B} + |\tilde f|_{0,\alpha;B} + |u^+|_{2,\alpha;G})$

and we thus obtain a bound independent of n. Thus, $v_n \in C^{2,\alpha}(\bar G)$ and is uniformly bounded, thus by Arzelà–Ascoli, we have that $\exists$ subsequence (not relabelled) s.t.

$v_n \to v^* \in C^{2,\alpha}(\bar G)$, convergence happens in $C^2(\bar G)$. Thus, v^* satisfies $\tilde L v^* = \tilde f$ in G.

Now, NTS: $v^* = \tilde v|_G$, $\tilde v \in C^{2,\alpha}(B) \cap C^0(\bar B)$.

By interior Schauder estimates, as in Thm 4.3, we have that up to a subsequence, $v_n \to v \in C^{2,\alpha}_{loc}(B)$ in $C^2(B)$. Additionally, $v \in C^0(\bar B)$ and $v_n \to v$ uniformly in B (apply WMP a priori estimate). Thus, $v \in C^{2,\alpha}(B) \cap C^0(\bar B)$ and $\tilde L v = \tilde f$ in B, $v = \tilde u$ on ∂B.

By uniqueness $\tilde v = v$ on B, and in particular $\tilde v|_G = v|_G = v^* \in C^{2,\alpha}(\overline{G \cap B})$. Hence, pulling back to Ω, $\exists$ nbhd U of $x_0 \in \partial\Omega$ (chosen arbitrarily) s.t. $u \in C^{2,\alpha}(\overline{U \cap \Omega})$, concluding the proof.

LECTURE 19

Thm 4.12 (Fredholm Alternative) Let $\alpha \in (0,1)$ and let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^{2,\alpha}$ domain. Let $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{\Omega})$ and $Lu \equiv a^{ij}D_{ij}u + b^i D_i u + cu$ h.n. strictly elliptic. Then either:

(i) the homogeneous problem
$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$
has a non-trivial solution $u \in C^{2,\alpha}(\overline{\Omega})$ OR

(ii) for any given $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$, the Dirichlet problem
$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$
has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.

Remark: ① This says that the sharp condition under which (ii) holds is not that $c \leq 0$, but that the homog. problem has only the zero solution.
[See example sheet for a case when $c > 0$ and still the homog. problem has only the zero solution]
② Failure of (i) is equivalent to the statement that uniqueness holds for solutions to the DP as in (ii). So the theorem can be seen as saying that if uniqueness holds (i.e. DP as in (ii) can have at most one solution), then there a solution exists.

Proof: It suffices to prove the theorem in the special case $\varphi \equiv 0$. (unique solvability of $Lu = f$ in Ω, $u = \varphi$ on $\partial\Omega$ for $u \in C^{2,\alpha}(\overline{\Omega})$ $\Leftrightarrow$ unique
" " $Lv = f - L\varphi$ in Ω, $v = 0$ $\partial\Omega$ for $v \in C^{2,\alpha}(\overline{\Omega})$)
So assume $\varphi \equiv 0$.
Choose constant $\sigma \geq \sup_{\Omega} c$ and let

$$L_\sigma u = Lu - \sigma u \equiv a^{ij}D_{ij}u + b^i D_i + \overbrace{(c-\sigma)}^{\leq 0}u$$

By Theorem 4.11, we know that
$L_\sigma : C_0^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$ is a bijection.
By Global Schauder estimate + max. principle estimate, $|u|_{2,\alpha;\overline{\Omega}} \leq C \cdot |L_\sigma u|_{0,\alpha;\overline{\Omega}}$ $\forall u \in C_0^{2,\alpha}(\overline{\Omega})$.
Equivalently, $|L_\sigma^{-1} f|_{2,\alpha;\overline{\Omega}} \leq C \cdot |f|_{0,\alpha;\overline{\Omega}}$ $\forall f \in C^{0,\alpha}(\overline{\Omega})$.
$L_\sigma^{-1} : C^{0,\alpha}(\overline{\Omega}) \to C_0^{2,\alpha}(\overline{\Omega})$ is a bdd linear operator.
The inclusion $I : C_0^{2,\alpha}(\overline{\Omega}) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ is compact by Arzelà-Ascoli.(*) So $T_\sigma \equiv I \circ L_\sigma^{-1} : C^{0,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$ is compact [i.e. if $(u_i)_i$ is bdded in $C^{0,\alpha}(\overline{\Omega})$, then $(T_\sigma(u_i))_i$ has a convergent subseq. in $C^{0,\alpha}(\overline{\Omega})$]. Hence so is σT_σ.
By the abstract Fredholm Alternative (last lecture), we have either
(i) $u + \sigma T_\sigma u = 0$ has a non-zero soln $u \in C^{0,\alpha}(\overline{\Omega})$, or
(ii) for any $f \in C^{0,\alpha}(\overline{\Omega})$, there is a unique function $u \in C^{2,\alpha}(\overline{\Omega})$ s.t. $u + \sigma T_\sigma u = L_\sigma^{-1} f$
Note that in either case, (since $T_\sigma u, L_\sigma^{-1} f \in C_0^{2,\alpha}(\overline{\Omega})$, and $u = -T u$ in case (i), and $u = L_\sigma^{-1} f - T_\sigma u$ in case ii), we have that $u \in C_0^{2,\alpha}(\overline{\Omega})$ automatically). Now just apply L_σ to both sides of the equation in both cases (i), (ii) $\square$

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§5. Quasilinear second order elliptic theory and the De Giorgi-Nash-Moser theory

Fix $\alpha \in (0,1)$, Ω a bdd $C^{2,\alpha}$ domain, $a^{ij}, b^i \in C^{0,\alpha}(\underset{x}{\overline{\Omega}} \times \underset{z}{\mathbb{R}} \times \underset{p}{\mathbb{R}^n}; \mathbb{R})$. Suppose $[a^{ij}(x,z,p)]_{ij}$ is positive definite for all $(x,z,p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$.
Consider the second-order quasilinear operator
$$Qu \equiv a^{ij}(x, u, Du) D_{ij} u + b(x, u, Du)$$
We are interested in the Dirichlet problem for Q, i.e. the question of solvability for $u \in C^{2,\alpha}(\overline{\Omega})$ of
$$\begin{cases} Qu = 0, & \text{in } \Omega \\ u = \varphi, & \text{on } \partial\Omega \end{cases}$$
for given $\varphi \in C^{2,\alpha}(\overline{\Omega})$.
To do this, we will rely on the following fixed point theorem.

Thm 5.1 (Leray-Schauder fixed pt. thm.): Let X be a Banach space, and $T : X \to X$ a continuous, compact operator (not assumed linear). Suppose there is a constant $M > 0$ for any $\sigma \in [0,1]$ and any $x \in X$ satisfying $x = \sigma T x$, we have that $\|x\| \leq M$ (M indep. of σ). Then $\exists x_0 \in X$ s.t. $x_0 = T x_0$, i.e. T has a fixed point.

(*) Compactness follows by exploiting boundary regularity (need at least C^1) and proceed similarly to the proof of the Global Schauder estimates (Thm 4.3).

LECTURE 20

$$Qu \equiv a^{ij}(x,u,Du)D_{ij}u + b(x,u,Du)$$

(DP) $\begin{cases} Qu = 0 \text{ in } \Omega, & \Omega \subset \mathbb{R}^n \text{ bdd}, C^{2,\alpha} \text{ domain}, \\ u = \varphi \text{ on } \partial\Omega & \varphi \in C^{2,\alpha}(\overline{\Omega}). \end{cases}$

We look for solution $u \in C^{2,\alpha}(\overline{\Omega})$.

To apply this result to (DP) above: We will take $X = C^{1,\beta}(\overline{\Omega})$ for some fixed $\beta \in (0,1)$. Define $T: C^{1,\beta}(\overline{\Omega}) \to C^{1,\beta}(\overline{\Omega})$ by setting $T(v) := u$ for any given $v \in C^{1,\beta}(\overline{\Omega})$, where u solves the linear Dirichlet problem:

$$\begin{cases} a^{ij}(x,v,Dv)D_{ij}u + b(x,v(x),Dv(x)) = 0 \text{ in } \Omega \\ u = \varphi \text{ on } \partial\Omega \end{cases}$$

Note for given $v \in C^{1,\beta}(\overline{\Omega})$, $x \mapsto a^{ij}(x,v(x),Dv(x))$, $x \mapsto b(x,v(x),Dv(x))$ are in $C^{0,\alpha\beta}(\overline{\Omega})$.
Also, $a^{ij}(x,v(x),Dv(x)) \geq \lambda_v \cdot |\xi|^2$ for some $\lambda_v > 0$.
By Thm 4.11, there is a unique $u \in C^{2,\alpha\beta}(\overline{\Omega}) \subseteq C^{1,\beta}(\overline{\Omega}) := X$ solving $(DP)_L$. So T is well defined. By the Schauder estimates, one can check that T is continuous and compact.
Note also that (DP) is equivalent to T having a fixed point, i.e. the existence of $v \in C^{1,\beta}(\overline{\Omega})$ satisfying $v = T(v)$. [such v automatically will be in $C^{2,\alpha\beta}(\overline{\Omega})$]. More generally; $v \in C^{1,\beta}(\overline{\Omega})$ satisfies $v = \sigma T(v)$ for some $\sigma \in [0,1] \iff$ $v \in C^{2,\alpha\beta}(\overline{\Omega})$, and solves $a^{ij}(x,v,Dv)D_{ij}v + \sigma b(x,v,Dv) = 0$ in Ω, $v = \sigma\varphi$ on $\partial\Omega$.
$\iff v \in C^{2,\alpha}(\overline{\Omega})$ and solves

$$\begin{cases} a^{ij}(x,v,Dv)D_{ij}v + \sigma b(x,v,Dv) = 0, & \text{in } \Omega \\ v = \sigma\varphi, & \text{on } \partial\Omega. \end{cases}$$

So by the abstract Leray-Schauder f.p. thm, if $\exists$ $M = M(n,\varphi,a^{ij},b,\Omega) > 0$ and some fixed $\beta = \beta(n,\varphi,a^{ij},b,\Omega) \in (0,1)$ s.t. $|u|_{1,\beta;\overline{\Omega}} \leq M$ whenever $u \in C^{2,\alpha}(\overline{\Omega})$ solves $a^{ij}(x,u,Du)D_{ij}u + \sigma b(x,u,Du) = 0$ in Ω, $u = \sigma\varphi$ on $\partial\Omega$ for some $\sigma \in [0,1]$, then (DP) has a solution in $C^{2,\alpha}(\overline{\Omega})$.

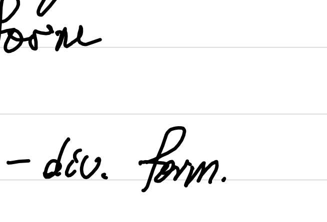
<u>Rmk:</u> Proving such a bound $|u|_{1,\beta;\Omega} \leq M$ generally requires additional hypotheses, e.g. in the case of <u>minimal surface equation</u> (see below), this requires the geometric condition that the domain Ω is <u>"mean convex"</u> (and $C^{2,\alpha}$).

Now let's consider operators Q arising as Euler-Lagrange operators associated with functionals of the form $\mathcal{F}(u) = \int_{\Omega} F(x,u,Du)\,dx$,

$$F: \underset{x}{\Omega} \times \underset{z}{\mathbb{R}} \times \underset{p}{\mathbb{R}^n} \to \mathbb{R}.$$

Exercise to check: E-L eqns i.e. $0 = \frac{d}{dt}\Big|_{t=0} \mathcal{F}(u+t\eta)$, $\eta = C_c^1(\Omega)$ has divergence structure which in non-divergence form is a Q as above.

Assume as a further simplification that $F(x,z,p) = F(p)$, i.e. F depends only on the "gradient variable" $p \in \mathbb{R}^n$.
A very important specific example (the prototypical quasilinear 2^{nd} order elliptic operator), is the case when $F(p) = \sqrt{1+|p|^2}$. So we have the area functional $\mathcal{F}(u) = \int_{\Omega} \sqrt{1+|Du|^2} \leftarrow$ area of graph



E-L eqn is the minimal surface equation:

$$D_i\left(\frac{D_i u}{\sqrt{1+|Du|^2}}\right) = 0, \leftarrow \text{div. form}$$

$$\left(\delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2}\right) = 0 \longleftarrow \text{non-div. form.}$$

In the generality of $F = F(p)$, the E-L equation is $D_i(F_{p_i}(Du))$, $F_{p_i}(p) = \partial/\partial p_i F(p)$.
$\uparrow$ div. form.
In non-divergence form $F_{p_i}F_{p_j}(Du)D_{ij}u = 0$. If the integrand is convex in p, then $a^{ij}(p) = F_{p_i p_j}(p)$ is elliptic. In the case of MSE:

$$a^{ij}(p) = \underbrace{\left(\delta_{ij} - \frac{p_i p_j}{1+|p|^2}\right)}_{\tilde{a}^{ij}(p)} \frac{1}{\sqrt{1+|p|^2}}$$

$[\tilde{a}^{ij}(p)]$ has eigenvalues $\underbrace{1,1,\ldots,1}_{(n-1)}, \frac{1}{1+|p|^2}$

is elliptic, but strictly elliptic only if $|Du|$ bdd in $\overline{\Omega}$.

We want a $C^{1,\beta}(\overline{\Omega})$ bound in $C^{2,\alpha}(\overline{\Omega})$ solutions to

$$\begin{cases} F_{p_i p_j}(Du)D_{ij}u = 0, & \text{in } \Omega \\ u = \sigma\varphi, & \text{on } \partial\Omega \end{cases}$$

By the WMP, $|u|_{0;\overline{\Omega}} \leq |\sigma\varphi|_{0;\Omega} \leq |\varphi|_{0;\Omega}$ (First easiest step).
Then we need (i) $|u|_{1;\overline{\Omega}} \leq M_1 = M_1(\varphi,F,\Omega)$.
(ii) $[Du]_{\beta;\overline{\Omega}} \leq M_2 = M_2(\varphi,F,\Omega)$.
For both of these, we derive and use the PDE satisfied by partial derivatives $w = D_k u$, $k \in \{1,\ldots,n\}$.
Typically, (i) will come from applying the max. principle and (ii) requires <u>De Giorgi-Nash-Moser</u>

Lecture 21

$\mathcal{F}(u) = \int_\Omega F(Du)\,dx$, E.g. $F(p) = \sqrt{1+|p|^2}$

$\rightsquigarrow$ Minimal surface equation.

E-L equation $\int_\Omega F_{p_i}(Du) D_i \eta = 0 \quad \forall \eta \in C_c^1(\Omega)$ ①

$(\Leftrightarrow D_i(F_{p_i}(Du)) = 0$ weakly in $\Omega)$.

To bound $|Du|_{0,\beta;\bar\Omega} \le M_1 = M_1(\varphi, F, \Omega)$

(ii) $|Du|_{0;\bar\Omega} \le M_2 \rightsquigarrow$ VERY PROBLEM DEPENDENT.

(iii) $[Du]_{\beta;\Omega} \le M_3$.

To do both (ii), (iii), we need the equation for partial derivatives $w = D_k u$, $k \in \{1, \dots, n\}$. Replace η with $D_k \eta$ in ①:

$$\int_\Omega F_{p_i}(Du) D_i D_k \eta = 0 \quad \forall \eta \in C_c^2(\Omega)$$

IBP wrt x_k: $\int_\Omega D_k(F_{p_i}(Du)) D_i \eta = 0$

$$\Longrightarrow \int_\Omega F_{p_i p_j}(Du) D_j w\, D_i \eta = 0$$

So w is a weak solution to

$D_i(F_{p_i p_j}(Du) \cdot D_j w) = 0$ in Ω. — ②

So if $\underset{\text{on } \bar\Omega}{|Du| \le M_2}$, then this is a uniformly elliptic equation in Ω.

Step (iii) follows [once we have step (ii)], by applying De Giorgi–Nash–Moser (DGNM) theory to equation ②. This theory in the interior says that if $w \in W^{1,2}$ is a weak solution to a divergence structure equation $D_i(a^{ij}(x) D_j w) = 0$ in Ω, with a^{ij} bdd meas. and strictly elliptic, then $\exists \beta \in (0,1)$ depending only on a bound on $\sum_{i,j} \|a^{ij}\|_{L^\infty(\Omega)}$ and the ellipticity constant, and dim n, s.t. solution $w \in C^{0,\beta}_{loc}(\Omega)$ (with an estimate).

Rmk: (1) DGNM theory extends to more general div. structure eqns which have lower order terms, as well as inhomogeneous terms on the r.h.s. under approp. assumptions. We will only present the theory for pure divergence form, homogeneous eqns as above.

(2) There is also global estimates, giving a bound on $|w|_{0,\beta;\bar\Omega}$ subject to appropriate boundary assumptions and that's what is really needed for the quasilinear applications. We will omit the bdry theory.

De Giorgi – Nash – Moser theory

We consider operators of the form

$Lu \equiv D_i(a^{ij} D_j u)$ on some open set $\Omega \subseteq \mathbb{R}^n$.

Hypothesis (H): (i) $a^{ij} \in L^\infty(\Omega)$ with $\|a^{ij}\|_{L^\infty(\Omega)} \le \Lambda$, Λ = a fixed constant and (ii) $a^{ij}(x)\xi_i \xi_j \ge \lambda|\xi|^2$, a.e. $x \in \Omega$, for some fixed constant $\lambda > 0$.

Def: A function $u \in W^{1,2}(\Omega)$ is a weak sub (super) solution to $Lu = 0$ in Ω if $\int_\Omega a^{ij} D_i u D_j v \le 0 \ (\ge 0)$ $\forall v \in W_0^{1,2}(\Omega)$ and $v \ge 0$.

Rmk: A $u \in W^{1,2}(\Omega)$ is a weak solution to $Lu = 0$ in Ω $\Leftrightarrow$ u is both a weak subsolution and a weak supersolution.

Thm 5.2: (Local boundedness of subsolutions). Suppose hyp. (H) holds. If $u \in W^{1,2}(\Omega)$ is a weak subsolution to $Lu = 0$ in Ω, then for any ball $B_{2R}(y) \subset \Omega$ and any $p > 1$.

$$\sup_{B_R(y)} u \le C \cdot R^{-n/p} \|u^+\|_{L^p(B_{2R}(y))}, \text{ where}$$

$C = C(n, \lambda, \Lambda, p)$.

Thm 5.3 (Weak Harnack inequality for non-neg. supersolutions). Suppose hyp. (H) holds. If $u \in W^{1,2}(\Omega)$ is a weak-supersolution to $Lu = 0$ in Ω, non-negative in $B_{4R}(y) \subset \Omega$, and if $p \in [1, \frac{n}{n-2})$, then

$$\inf_{B_R(y)} u \ge C \cdot R^{-n/p} \|u\|_{L^p(B_{2R}(y))}, \quad C = C(n, \lambda, \Lambda, p).$$

Corollary 5.4 (Harnack inequality for non-negative solutions). Hyp(H) holds. If $u \in W^{1,2}(\Omega)$ is a non-neg. weak solution to $Lu = 0$ in Ω, then for any subdomain $\Omega_1 \subset\subset \Omega$, we have

$$\sup_{\Omega_1} u \le C \cdot \inf_{\Omega_1} u, \text{ where}$$

$C = C(n, \lambda, \Lambda, \Omega_1, \Omega)$.

Proof: Just pick some $p \in (1, n/n-2)$, and apply Thms 5.2 and 5.3 to get the Harnack ineq. for balls. Then use the same argument as in the case of non-negative harmonic functions to extend it to domains $\Omega_1 \subset\subset \Omega$.

Theorem 5.5: (Hölder continuity). Let hyp(H) hold, and suppose that $u \in W^{1,2}(\Omega)$ is a weak solution to $Lu = 0$ in Ω. Then u is (a.e.) locally Hölder cts in Ω. Moreover, we have the estimates: for any ball $B_R(y) \subset \Omega$,

(i) for any $r \in (0, R]$, we have

$$\operatorname{osc}_{B_r(y)} u \le C \cdot \left(\frac{r}{R}\right)^\mu \cdot \operatorname{osc}_{B_R(y)} u; \quad (\operatorname{osc} = \sup - \inf)$$

(ii) $u \in C^{0,\mu}(\Omega)$, and $R^\mu [u]_{\mu; B_{R/4}(y)} \le C \cdot \sup_{B_R(y)} |u|$

$$C = C(n, \lambda, \Lambda), \quad \mu = \mu(n, \lambda, \Lambda) \in (0,1).$$

LECTURE 22

Thm 5.5: $Lu = 0$, $u \in W^{1,2}(\Omega)$ $\forall B_R(y) \subset \Omega$,

(i) $\forall r \in (0, R]$, $\operatorname*{osc}_{B_r(y)} u \le C \cdot \left(\frac{r}{R}\right)^{\mu} \operatorname*{osc}_{B_R(y)} u$.

(ii) $R^{\mu}[u]_{\mu; B_{R/4}(y)} \le C \cdot \sup_{B_R(y)} |u|$, $C = C(n, \lambda, \Lambda)$

Proof: First check (i) $\Rightarrow$ (ii): $x, z \in B_{\frac{R}{4}}(y)$, $x \ne z$. Let $d = 5/4 |x - z|$.

$d \le \frac{5}{4} \cdot R/2 = \frac{5}{8} R$, $B_{\frac{5R}{8}}(x) \subset B_R(y)$.

$$|u(x) - u(z)| \le \operatorname*{osc}_{B_d(x)} u \le C \cdot \left(\frac{d}{\frac{5R}{8}}\right)^{\mu} \underbrace{\operatorname*{osc}_{B_{\frac{5R}{8}}(x)} u}_{\le \operatorname*{osc}_{B_R(y)} u}$$

(by (i)) $\Rightarrow$ $R^{\mu} \frac{|u(x) - u(z)|}{|x - z|^{\mu}} \le C \cdot \sup_{B_R(y)} |u|$

$\Longrightarrow$ $R^{\mu}[u]_{\mu; B_{R/4}(y)} \le C \cdot \sup_{B_R(y)} |u|$.

To see (i), we'll use Thm 5.3 (Note also that $|u|$ is locally bdd by Thm 5.2 applied to u and $-u$).

Suppose $r \le R/4$. Set $M_1 = \sup_{B_r(y)} u$, $m_1 = \inf_{B_r(y)} u$, $M_4 = \sup_{B_{4r}(y)} u$, $m_4 = \inf_{B_{4r}(y)} u$.

(want to establish $(M_1 - m_1) \le \gamma (M_4 - m_4)$, $\gamma \in (0,1)$ and iterate).

Then $M_4 - u$ and $u - m_4$ are both non-negative in $B_{4r}(y)$, and satisfy $L(M_4 - u) = 0$, $L(u - m_4) = 0$. So by Thm 5.3 with $p = 1$,

$$r^{-n} \int_{B_{2r}} (M_4 - u) \le C \cdot \inf_{B_r(y)} (M_4 - u)$$

$$= C \cdot (M_4 - M_1) \quad — \text{①}$$

$$r^{-n} \int_{B_{2r}} (u - m_4) \le C \cdot \inf_{B_r(y)} (u - m_4) = C \cdot (m_1 - m_4) \quad — \text{②}$$

① + ② $\Longrightarrow$ $(M_4 - m_4) \le C \cdot \left((M_4 - m_4) - (M_1 - m_1)\right)$

$\Longrightarrow$ $(M_1 - m_1) \le \left(\frac{C-1}{C}\right) \cdot (M_4 - m_4)$

$\gamma = \frac{C-1}{C} < 1$.

$\Longrightarrow$ $\operatorname*{osc}_{B_r(y)} u \le \gamma \operatorname*{osc}_{B_{4r}(y)} u$, $\gamma \in (0,1)$

Iterate this, starting with $r = R/4$:

$$\operatorname*{osc}_{B_{4^{-(j+1)}R}} u \le \gamma^j \operatorname*{osc}_{B_{R/4}} u, \quad j = 0, 1, 2, \ldots$$

Given any $r \in (0, R/4]$, $\exists$ (unique) j s.t. $4^{-(j+1)} R < r \le 4^{-j} R$. So

$$\operatorname*{osc}_{B_r(y)} u \le \operatorname*{osc}_{B_{4^{-j}R}(y)} u \le \gamma^{j-1} \operatorname*{osc}_{B_{R/4}(y)} u$$

$$= \gamma^{-1} \cdot 4^{\left(\frac{\log \gamma}{\log 4}\right) j} \operatorname*{osc}_{B_{R/4}(y)} u, \quad \text{let } \mu = \frac{\log \gamma}{\log (1/4)} \in (0, \infty).$$

$$= \gamma^{-1} \cdot 4^{-j\mu} \operatorname*{osc}_{B_{R/4}(y)} u$$

$$\Longrightarrow \operatorname*{osc}_{B_r(y)} u \le C \cdot \left(\frac{r}{R}\right)^{\mu} \operatorname*{osc}_{B_{R/4}(y)} u.$$

If $R/4 \le r \le R$, then $\operatorname*{osc}_{B_r(y)} u \le \operatorname*{osc}_{B_R(y)} u$

$$= 4^{\mu} \cdot \frac{1}{4^{\mu}} \operatorname*{osc}_{B_R(y)} u \le 4^{\mu} \left(\frac{r}{R}\right)^{\mu} \operatorname*{osc}_{B_R(y)} u$$

Proof of Thm 5.2: We are assuming $u \in W^{1,2}(\Omega)$ is a weak subsolution i.e. $\int_{\Omega} a^{ij} D_i u D_j v \le 0$

$\forall v \in W^{1,2}(\Omega)$, $v \ge 0$. Then $u^+ = \max\{u, 0\}$ is also a subsolution (Ex. 4). So wlog we can assume that u is non-negative.

It suffices to prove the theorem assuming in fact that $y \ge \varepsilon$ for arbitrary $\varepsilon > 0$. (The general case $u \ge 0$ then follows by applying the conclusion to $u + \varepsilon$, and letting $\varepsilon \to 0$).

By considering $\tilde{u}(x) = u(y + Rx)$, we may also assume $y = 0$, $R = 1$. Let $\beta > 0$ and let $v_k = \min\{u^{\beta}, ku\}$ for suff. large k.

<u>Claim:</u> $v_k \in W^{1,2}(B)$, with

$$D v_k(x) = \begin{cases} \beta u^{\beta - 1} Du(x) & \text{if } x \in \Omega_k \equiv \{x \in B : u^{\beta} \le k u(x)\} \\ k Du & \text{if } x \in B \setminus \Omega_k. \end{cases}$$

$1 \le k u^{1-\beta}(x)$

To see this, note that if $\beta \le 1$, $\Omega_k = B$ and $v_k = u^{\beta}$ for suff. large k (since $u \ge \varepsilon$).

When $\beta > 1$, then $v_k = u \min\{u^{\beta - 1}, k\}$
$= u \cdot g(\omega_k)$, where $\omega_k = \min\{u, k^{\frac{1}{\beta - 1}}\}$
$g(t) = t^{\beta - 1}$.

Since $\varepsilon \le \omega_k \le k^{\frac{1}{\beta - 1}}$, and g and g' are bdd in $[\varepsilon, k^{\frac{1}{\beta - 1}}]$, we have that $g(\omega_k) \in W^{1,2}(B)$ with $D(g(\omega_k)) = g'(\omega_k) \cdot D\omega_k$ (by standard facts about Sobolev functions). <u>Claim follows.</u>
($\omega_k \in W^{1,2}$, being min of Sobolev f^n & constant).

Fix $\eta \in C_c^1(B)$ and take $v = v_k \eta^2$ in the inequality $\int_B a^{ij} D_i u D_j v \le 0$.

$$\beta \int_{\Omega_k} a^{ij} D_i u \cdot u^{\beta - 1} D_j u \, \eta^2 + k \int_{B \setminus \Omega_k} a^{ij} D_i u \cdot D_j u \, \eta^2$$

$$\le -2 \int_{\Omega_k} a^{ij} D_i u \cdot u^{\beta} \eta D_j \eta - 2k \int_{B \setminus \Omega_k} a^{ij} D_i u \cdot u \, \eta \, D_j \eta$$

By ellipticity and bounds $\|a^{ij}\|_{L^{\infty}(B)} \le \Lambda$, this gives:

$$\beta \int_{\Omega_k} |Du|^2 \cdot u^{\beta - 1} \cdot \eta^2 + \lambda k \int_{B \setminus \Omega_k} |Du|^2 \eta^2$$

$$\le \frac{2\Lambda}{\lambda} \underbrace{\int_{\Omega_k} |Du| u^{\beta} \cdot \eta \cdot |D\eta|} + \frac{2\Lambda k}{\lambda} \cdot \int_{B \setminus \Omega_k} |Du| \cdot u \cdot \eta \cdot |D\eta|$$

$$= \int_{\Omega_k} |Du| u^{\frac{\beta - 1}{2}} u^{\frac{\beta + 1}{2}} \eta |D\eta|$$

Young's ineq. with epsilon

$$\le \frac{\beta}{2} \int_{\Omega_k} |Du|^2 u^{\beta - 1} |\eta|^2$$

$$+ \frac{C}{\beta} \int_{\Omega_k} u^{\beta + 1} |D\eta|^2$$

$$\Longrightarrow \beta/2 \int_{\Omega_k} |Du|^2 u^{\beta - 1} \eta^2 + k/2 \int_{B \setminus \Omega_k} |Du|^2 \eta^2$$

$$\le C/\beta \int_{\Omega_k} u^{\beta + 1} |D\eta|^2 + ck \underbrace{\int_{B \setminus \Omega_k} u^2 \cdot |D\eta|^2}$$

$ku \le u^{\beta}$

LECTURE 23

Proof of Thm 5.2 (Cont'd):

$\forall \beta > 0$, $\forall k$ large,

$$\beta/2 \int_{\Omega_k} |Du|^2 u^{\beta-1} \eta^2 + \frac{k}{2} \int_{B\setminus\Omega_k} |Du|^2 \eta^2$$

$$\le C/\beta \int_{\Omega_k} u^{\beta+1} |D\eta|^2 + Ck \int_{B\setminus\Omega_k} u^2 |D\eta|^2$$

where $C = C(\frac{\Lambda}{\lambda})$, $\Omega_k = \{x \in B : u^\beta(x) \le k u(x)\}$

$$\Rightarrow \quad \beta/2 \int_{\Omega_k} |Du|^2 u^{\beta-1} \eta^2 \le C/\beta \int_{\Omega_k} u^{\beta+1} |D\eta|^2$$

$$+ C \int_{B\setminus\Omega_k} u^{\beta+1} |D\eta|^2$$

Note that $1_{\Omega_k} \to 1_B$ pointwise in B $(u > \varepsilon)$,

Assuming $\int_B u^{\beta+1} |D\eta|^2 < \infty$, we can let $k \to \infty$, to deduce that

$$\beta/2 \int_B |Du|^2 u^{\beta-1} \eta^2 \le C/\beta \int_B u^{\beta+1} |D\eta|^2$$

let $\alpha = \beta + 1$, $\int_B |Du|^2 u^{\alpha-2} \eta^2 \overset{⊛}{\le} \frac{C}{(\alpha-1)^2} \int_B u^\alpha |D\eta|^2$

holds for any $\alpha > 1$, where $C = C(\frac{\Lambda}{\lambda})$ provided $\int_B u^\alpha |D\eta|^2 < \infty$.

$$D(u^{\frac{\alpha}{2}} \eta) = \alpha/2 \, u^{\frac{\alpha-2}{2}} Du \cdot \eta + u^{\alpha/2} \cdot D\eta.$$

$$\Rightarrow \quad \int_B |D(u^{\alpha/2} \eta)|^2 \le \frac{C \cdot \alpha^2}{(\alpha-1)^2} \int u^{\alpha-1} |Du|^2 \eta^2$$

$$+ 2 \int_B u^\alpha \cdot |D\eta|^2$$

$$\overset{⊛}{\le} \frac{C\alpha^2}{(\alpha-1)^2} \int_B u^\alpha |D\eta|^2$$

Recall the Sobolev inequality: $(f \in W_0^{1,2}(B))$

$$\|f\|_{L^{2\sigma}(B)} \le C \cdot \|Df\|_{L^2(B)}, \quad \sigma = \begin{cases} \frac{n}{n-2} & \text{if } n \ge 3. \\ \text{any fixed } \# & \text{if } n = 2 \end{cases}$$

$C = C(n)$.

Using this with $f = u^{\alpha/2} \eta$, we get from the previous line that,

$$\left(\int_B (u^{\alpha\sigma} \eta^{2\sigma}) \right)^{1/\sigma\alpha} \le \left(C \cdot \frac{\alpha^2}{(\alpha-1)^2} \right)^{1/\alpha} \left(\int_B u^\alpha |D\eta|^2 \right)^{1/\alpha},$$

subject to $\int_B u^\alpha |D\eta|^2 < \infty$

Given $0 < r' < r < 1$, choose η so that $\eta \in C_c^1(B)$, $\eta \equiv 1$ on $B_{r'}$ and $\eta \equiv 0$ in $B \setminus B_r$ and $|D\eta| \le \frac{2}{r - r'}$

$$\text{So} \quad \left(\int_{B_{r'}} u^{\alpha\sigma} \right)^{\frac{1}{\alpha\sigma}} \le \left(\frac{C\alpha^2}{(\alpha-1)^2} \right)^{1/\alpha} \frac{1}{(r-r')^{2/\alpha}} \left(\int_{B_r} u^\alpha \right)^{1/\alpha}$$

Let $r_j = 1/2 + 1/2^{j+1}$, and take $r = r_{j-1}$, $r' = r_j$ as well as $\alpha = p \cdot \sigma^{j-1}$ for any $p > 1$, for $j \ge 0$.

Since $g_\alpha = \alpha/\alpha - 1 = 1 + \frac{1}{\alpha - 1}$ is decreasing in α, we have $g(p\sigma^j) \le g(p) = p/p-1$

$$\left(\int_{B_{r_j}} u^{p\sigma^j} \right)^{1/p\sigma^j} \le C^{\frac{1}{p\sigma^{j-1}}} \cdot 2^{\frac{2(j+1)}{p\sigma^{j-1}}} \left(\int_{B_{r_{j-1}}} u^{p\sigma^{j-1}} \right)^{\frac{1}{p\sigma^{j-1}}}$$

$$C = C(p, \tfrac{\Lambda}{\lambda})$$

$j = 1, 2, \ldots$

Iterating this gives:

$$\left(\int_{B_{r_j}} u^{p\sigma^j} \right)^{\frac{1}{p\sigma^j}} \le C^{\sum_{j=1}^\infty \frac{1}{p\sigma^{j-1}}} \cdot 2^{\frac{2}{p} \sum_{j=1}^\infty \frac{j}{p\sigma^{j-1}}} \cdot \left(\int_{B_1} u^p \right)^{1/p}$$

$$= C \cdot \|u\|_{L^p(B)}$$

let $j \to \infty \implies \sup_{B_{1/2}} u \le C \cdot \|u\|_{L^p(B)}$,

$C = C(n, \frac{\Lambda}{\lambda}, p)$, $p > 1$ $\square$

The iteration technique used above to prove the theorem is called the <u>Moser iteration</u>.

<u>Remark:</u> Using the case $p > 1$ (in fact the case $p = 2$) of the theorem (proved), it is possible to extend to all $p > 0$ (with $C > 0$ depending on p). See Ex Sheet 4.

It remains to prove thm 5.3 (weak Harnack inequality). For this we need the following first:

<u>Lemma (John-Nirenberg):</u> Let $B = B_1(0) \subset \mathbb{R}^n$, and let $u \in W^{1,1}(B)$. Suppose that there is a $M > 0$ s.t. $\rho^{1-n} \int_{B_\rho(y) \cap B} |Du| \le M < \infty$ for any ball $B_\rho(y)$.

Then, there exists $p_0 = p_0(n)$ and $C = C(n)$ s.t.

$$\int_B e^{\frac{p_0}{M}|u - u_B|} \le C, \quad \text{where}$$

$u_B = 1/\mathcal{L}^n(B) \int_B u$

<u>Proof:</u> Omitted. See G&T, Ch 7.

<u>Proof of Thm 5.3:</u> $\int_{B_4} a^{ij} D_i u D_j v \ge 0$,

w.l.o.g.

Assume $R = 1$, $y = 0$, also $u > \varepsilon$.

$w = 1/u$, $\implies \int_{B_4} a^{ij} D_i w D_j v \le -2\lambda \int \frac{|Du|^2}{u^2} v \le 0$

(put $w^2\varphi$ for test function)

(using ellipticity, check!)

So by thm 5.2, $\sup_{B_1} w \le C \cdot \left(\int_{B_2} w^p \right)^{1/p}$

$$\Rightarrow \quad \inf_{B_1} u \ge C \cdot \left(\int_{B_2} u^{-p} \right)^{-1/p}$$

$$= C \cdot \left(\int_{B_2} u^p \right)^{1/p} \cdot \left[\left(\int_{B_2} u^p \right) \left(\int_{B_2} u^{-p} \right) \right]^{-1/p}.$$

LECTURE 24

Proof of Thm 5.3 (continued)

WLOG $R=1$, $y=0$, $u \geq \varepsilon > 0$ (else replace u with $u+\varepsilon$, then let $\varepsilon \downarrow 0$).

① $-\int_{B_4} a^{ij} D_i u D_j v \geq 0 \quad \forall\, v \in W_0^{1,2}(B_4),\ v \geq 0.$

Let $w = 1/u$, $w \in W^{1,2}(B_4)$ since $u \geq \varepsilon$. $D_i u = -\frac{1}{w^2} D_i w$; also replace v with $w^2 v$.

$$-\int a^{ij} \frac{D_i w}{w^2} D_j v - \int a^{ij} \frac{D_i w}{w^2} w D_j w \geq 0$$

$\forall\, v \in C_c^1(B_4)$, $v \geq 0$.

$$\int a^{ij} D_i w D_j v \leq -2\lambda \int \frac{|Dw|^2}{w^2} v \leq 0,$$

So w is a non-negative weak sub-solution.

So by Thm 5.2 (+ remark at end of Thm 5.2) $\sup_{B_1} w \leq C \cdot \left(\int_{B_3} w^p\right)^{1/p} < \infty$ $\forall\, p \in (0, 2]$.

$$\Longrightarrow \inf_{B_1} u \geq C \cdot \left(\int_{B_3} u^{-p}\right)^{-1/p} = C \left(\int_{B_3} u^p\right)^{1/p} \left[\left(\int_{B_3} u^{-p}\right)\left(\int_{B_3} u^p\right)\right]^{-1/p}$$

$C = C(n, \Lambda/\lambda, p)$.

Claim: $\exists\, p_0 = p_0(n, \Lambda/\lambda) > 0$ and $C = C(n, \Lambda/\lambda)$ s.t.

$$\left(\int_{B_3} u^{-p_0}\right)\left(\int_{B_3} u^{p_0}\right) \leq C$$

This will prove the theorem for $p = p_0$, and hence (by Hölder's inequality) for any $p \in (0, p_0]$.

Proof of claim: We rely on John-Nirenberg lemma.

Let $w = \log u - \frac{1}{|B_3|}\int_{B_3} \log u$ (since $u \geq \varepsilon$, $w \in W^{1,2}(B_3)$).

$D_i w = \frac{D_i u}{u}$, using ① with $u^{-1} v$ in place of v:

$$\int a^{ij} \cancel{u} D_i w\, \cancel{u^{-1}} D_j v - \int a^{ij} \cancel{u} D_i w \frac{1}{u^2} D_j u \cdot v \geq 0$$

$$\Longrightarrow \int a^{ij} D_i w D_j v \geq \int a^{ij} D_i w D_j w\, v \geq \lambda \int |Dw|^2 v$$

using $|a^{ij}| \leq \Lambda$, replacing v with v^2;

$$\int |Dw|^2 v^2 \leq \frac{2\Lambda}{\lambda} \int |Dw| \cdot |Dv| v$$

(replace v with v^2) $\forall\, v \in C_c^1(B_4)$, $v \geq 0$.

$$\leq \frac{1}{2}\int |Dw|^2 v^2 + \frac{\Lambda}{2\lambda}\int |Dv|^2 \quad ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$$

$$\Longrightarrow \text{②} \quad \int |Dw|^2 v^2 \leq \frac{\Lambda}{\lambda} \int |Dv|^2 \quad v \in C_c^1(B_4),\ v \geq 0.$$

If $B_{7/6\rho}(y) \subset B_4$, then we can choose v in ② $v \in C_c^1(B_{7/6\rho}(y))$, $v \equiv 1$ on $B_\rho(y)$ and $|Dv| \leq 12/\rho$, $v \geq 0$

$$\Longrightarrow \int_{B_\rho(y)} |Dw|^2 \leq C\rho^{n-2} \Longrightarrow \int_{B_\rho(y)} |Dw| \leq \left(\int_{B_\rho(y)} |Dw|^2\right)^{1/2} \times |B_\rho(y)|^{1/2} \leq C \cdot \rho^{n-1}.$$

We need to check that for <u>any</u> ball $B_\rho(y)$,

$$\int_{B_\rho(y) \cap B_3} |Dw| \leq C \cdot \rho^{n-1} \quad (**)$$

If $\rho \geq 1/4$, then $\int_{B_\rho(y) \cap B_3} |Dw| \leq \int_{B_3} |Dw| \leq C \leq C \cdot \rho^{n-1}$

For $\rho \in (0, 1/4)$, $\int_{B_\rho(y) \cap B_3} |Dw|$

if $B_\rho(y) \cap B_3 \neq \emptyset$

$\Rightarrow B_{7/6\rho}(y) \subset B_4$, so by again,

$$\int_{B_\rho(y) \cap B_3} |Dw| \leq \int_{B_\rho(y)} |Dw| \leq C \cdot \rho^{n-1}$$

So by John-Nirenberg $\int e^{p_0 |w|} \leq C$, for $p_0 = p_0(n, \frac{\Lambda}{\lambda})$, $C = C(n)$.

Note: $\int_{B_3} w = 0$.

So $\left(\int_{B_3} u^{-p_0}\right) \cdot \left(\int_{B_3} u^{p_0}\right) \leq \left(\int_{B_3} e^{-p_0 w}\right)\left(\int_{B_3} e^{p_0 w}\right) \leq C^2$, so the claim holds, and hence the theorem for $p \in (0, p_0]$.

To prove the theorem for $p \in (p_0, n/(n-2))$, it suffices to show that given such p,

$$\left(\int_{B_2} u^p\right)^{1/p} \leq C \cdot \left(\int_{B_3} u^{p'}\right)^{1/p'}, \text{ for some}$$

$0 < p' \leq p_0$, $p' = p'(p, n, \frac{\Lambda}{\lambda})$, $C = C(n, \frac{\Lambda}{\lambda})$.

To see this, let $\beta > 0$, take $v = u^{-\beta}\eta^2$ in $\int a^{ij} D_i u D_j v \geq 0$. By the exact same steps as in Thm 5.2:

$$\beta/2 \int u^{-\beta-1} |Du|^2 \eta^2 \leq \frac{C}{\beta} \int u^{1-\beta} |D\eta|^2$$

Letting $\gamma = 1-\beta$, and assuming $\gamma \in (0, 1)$, (i.e. $\beta \in (0,1)$), this gives us: $\int u^{\gamma-2} \cdot |Du|^2 \eta^2 \leq \frac{C}{(1-\gamma)^2} \int u^\gamma |D\eta|^2$

Sobolev: $\Longrightarrow$

$$(***) \quad \left(\int_{B_{r'}} u^{\sigma\gamma}\right)^{\frac{1}{\sigma\gamma}} \leq \left(\frac{C\gamma^2}{(1-\gamma)^2}\right)^{1/\gamma} \frac{1}{(r-r')^{2/\gamma}} \times \left(\int_{B_r} u^\gamma\right)^{1/\gamma},$$

$0 < r' < r < 4$, $\sigma = n/(n-2)$.

Given $p \in (p_0, \sigma)$, choose $N = N(n, \frac{\Lambda}{\lambda}, p) \in \mathbb{N}$ s.t. $\sigma^{-N} p \leq p_0$.

In (***), take $\gamma = \sigma^{-j-1} p$, $r' = 2 + j/N$, $r = 2 + \frac{(j+1)}{N}$, $j = 0, 1, 2, \dots, N-1$

$$\Longrightarrow \left(\int_{B_{2+j/N}} u^{\sigma^{-j} p}\right)^{\frac{1}{\sigma^{-j} p}} \leq \left(\frac{C\sigma^{-j-1} p}{1-\sigma^{-j-1} p}\right)^{\frac{2}{\sigma^{-j-1} p}} \cdot N^{\frac{2}{\sigma^{-j-1} p}} \times \left(\int_{B_{2+\frac{j+1}{N}}} u^{\sigma^{-j-1} p}\right)^{\frac{1}{\sigma^{-j-1} p}}$$

$j = 0, 1, 2, \dots, N-1$.

$$\Longrightarrow \left(\int_{B_2} u^p\right)^{1/p} \leq C \cdot \left(\int_{B_3} u^{\sigma^{-N} p}\right)^{\frac{1}{\sigma^{-N} p}} \qquad \square$$